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ANDERSON-DARLING AND CRAMÉR-VON MISES BASED GOODNESS-OF-FIT TESTS FOR THE WEIBULL DISTRIBUTION WITH KNOWN SHAPE USING NORMALIZED SPACINGS

THESIS

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THESIS

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Eric William Frisco

Table of Contents

List of Figures

Figure		Page

List of Tables

Table			Page

AFIT/GOR/ENC/98M-01

Abstract

Two new goodness-of-fit tests are developed for the three-parameter Weibull distribution with known shape parameter. These procedures eliminate the need for estimating the unknown location and scale parameters prior to initiating the tests and are easily adapted for censored data. This is accomplished by employing the Anderson-Darling A_s^2 and Cramér-von Mises W_S^2 statistics based on the normalized spacings of the sample data. Critical values of A_s^2 and W_s^2 are obtained for common significance levels by large Monte Carlo simulations for shapes of 0.5(0.5)4.0 and sample sizes of 5(5)40 with up to 40% censoring (type II) from the left and/or right. An extensive Monte Carlo power study is also conducted to compare the two tests with each other and with their prominent competitors. The competitors include another spacings test, Z^*, and the modified Kolmogorov-Smirnov (KS), Cramér-von Mises (W^2) and Anderson-Darling (A^2) EDF tests. The power results indicate that no one test is superior in all situations. When the alternatives considered are tested against a skewed Weibull null distribution, A_s^2 and W_s^2 achieve considerably higher power than the other EDF tests, but do not perform as well as Z^*. On the other hand, when the null distribution is symmetric, Z^* loses all of its power, while A_s^2 and W_s^2 yield power comparable to the other EDF tests. Results also show A_s^2 generally outperforms W_s^2, and for these reasons, A_s^2 is the preferred test for the three-parameter Weibull with known shape.

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I. INTRODUCTION

1.1 Motivation

The United States Air Force relies on a variety of complex weapon systems to complete its mission. With decreasing budgets and increasing costs of new technologies, the cost effectiveness of current and new systems is of great importance and interest. System reliability is a critical factor in such evaluations. However, characterizing a complex system and/or its components in terms of their expected failure times, for instance, is often impossible with purely analytical approaches due to the stochastic nature of the problem. For this reason, computer simulations employing statistical failure models are used extensively. A statistical failure model is a probability distribution which attempts to mathematically describe the lifetime of a material, a structure, or a device. In typical reliability and life-span analyses, computations within a simulation are based on the assumption that data collected through experimentation and observation follow a particular theoretical lifetime distribution. Thus, the selection of a model which adequately "fits" the data is crucial for valid results.

Goodness-of-fit tests provide statistical procedures to assess the validity of model and data correspondence. These tests check the likelihood that the observed data is a sample of the population defined by the hypothesized distribution. In this

study, goodness-of-fit statistics based on the normalized sample spacings are applied to the Weibull distribution with known shape parameter.

1.2 Weibull Distribution

Although there are many statistical distributions which can describe the lifetime of a component or system, this thesis only considers the Weibull distribution. The Weibull distribution is popular as a life-testing model and for many other applications which require a skewed distribution (3:207). Perhaps the most desirable property of the Weibull distribution is its flexibility. Variation of its location, scale, and shape parameters allows it to take on an infinite number of geometries; it can be skewed-left, symmetric, or skewed-right, as seen in Figure 1.1. This flexibility allows the Weibull to generalize many sets of data.

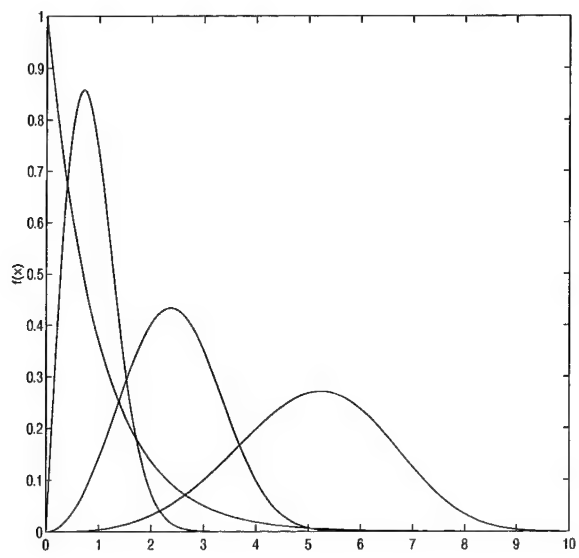

Figure 1.1 A Variety of Weibull Distributions

1.3 Overview of Statistical Tests

The objective of a statistical test is to evaluate a hypothesis concerning the values of one or more population parameters. All statistical tests of hypotheses work in the same way and are composed of the same essential elements:

- Null hypothesis, H_0
- Alternate hypothesis, H_a
- Test statistic
- Rejection region

Generally, the null hypothesis is the hypothesis we seek to disprove based on the information from the sample. Conversely, the alternate hypothesis is usually the hypothesis we seek to support. The test statistic is computed from the sample data and is designed so that if the null hypothesis is true, the distribution of the test statistic will be known. The rejection region specifies the range of values of the test statistic which result in rejection of the null hypothesis. The acceptance region specifies the range of values of the test statistic which result in failure to reject the null hypothesis. The number that is the boundary between the rejection region and the acceptance region is called the critical value of the test statistic. The statistical decision between the two hypotheses is determined by comparing the value of the test statistic with the critical value of the test statistic. Test statistics larger than their critical value result in rejection of the null hypothesis.

Decisions made concerning the null hypothesis can result in two different types of errors:

- Type I - H_0 is rejected when H_0 is true.
- Type II - H_0 is accepted when H_a is true.

The probabilities of type I and type II errors are denoted by α and β, respectively.

Thus, α and β measure the risks associated with making an erroneous decision. As such, they provide a practical way to measure the goodness of a test (31:412-413). The risk of committing a type I error is often called the significance level of the test. Another measure of the goodness of a test is the power of the test. Denoted by (1-β), it is the probability of rejecting the null hypothesis, H_0, when a particular alternate hypothesis is true. This measure will be critical in evaluating the usefulness of the tests developed in this research effort.

1.4 Goodness-of-fit Tests

Statistical hypothesis tests which measure how well a set of sample data agrees with a hypothesized distribution are termed goodness-of-fit tests. Though unlike most hypothesis tests, the goal of a goodness-of-fit test is to fail to reject the null hypothesis. In other words, the null hypothesis is the hypothesis we seek to support. As a result, these tests provide helpful guidance for evaluating the suitability of statistical models (4:375). For example, suppose sample lifetime data is collected for a particular component or system. Generally, the first step in selecting a particular distribution with which to model the data is to decide what families (e.g., exponential, normal, gamma, or Weibull) appear to be appropriate on the basis of their shapes without worrying (yet) about the specific parameter values for these families. Whenever possible, prior knowledge about the system under study should be used to select a modeling distribution, or to at least rule out some distributions (17:356-358). Once a statistical family has been selected, the distribution parameters, if not specified, are typically estimated from the sample data. A goodness-of-fit test is then conducted to see if the observed data could be a sample from the hypothesized distribution function. In this case, if a goodness-of-fit test indicates that the observed data fits the proposed statistical failure model (lifetime distribution), the hypothesized distribution can be used to predict characteristics of the sampled

component or system, such as the expected failure time or the probability that the component will still work at some time t.

Some of the most commonly used classical goodness-of-fit tests include the following:

- Chi-square (χ^2)
- Kolmogorov-Smirnov (K-S)
- Cramér-von Mises (W^2)
- Anderson-Darling (A^2)

These tests, and several others, are discussed in more detail in the next chapter.

1.5 Problem Statement

When modeling a random variable of a system using the Weibull distribution, the shape parameter is often known, while the location and scale parameters are unknown. Harter and Moore (15) summarize values of shapes for various Weibull applications. The majority of the goodness-of-fit tests available for this particular family of the Weibull distribution require estimation of the unknown location and scale parameters. Parameter estimation adds to the complexity of these tests because the techniques typically implemented (e.g. maximum likelihood, minimum distance, etc.) are computationally intensive. Furthermore, each time the goodness-of-fit test rejects the null hypothesis, the parameter estimation routine typically must be employed again for the new hypothesized distribution, as shown in Figure 1.2. The goal of this research is to develop more powerful omnibus goodness-of-fit tests for the Weibull distribution with the benefit of eliminating the need for location and scale parameter estimation when the shape parameter is known. This will be accomplished by using modifications of the Anderson-Darling and Cramér-von Mises statistics based on normalized sample spacings. Of course, once a distribution is

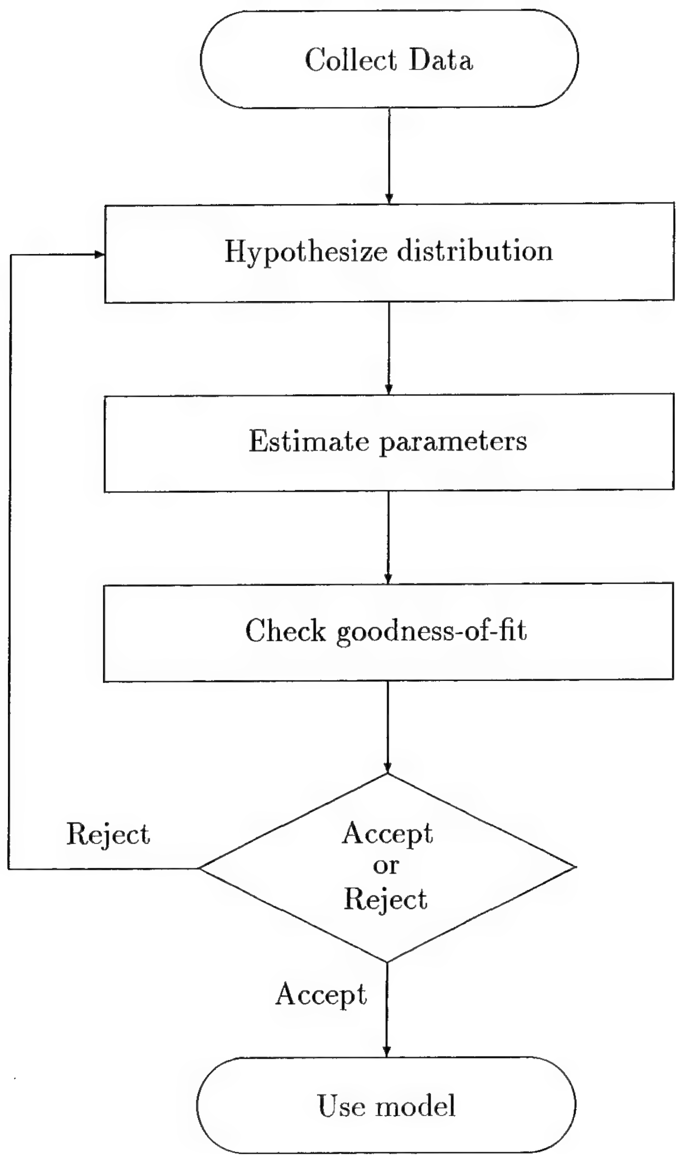

Figure 1.2 Typical Input Modeling Process

accepted as an adequate model, the remaining unknown parameters must then be estimated for use in a simulation model, as illustrated in Figure 1.3. By requiring the estimation of the location and scale parameters only once, the procedures developed in this thesis will dramatically reduce computational requirements.

1.6 Objectives/Scope

The objectives of this thesis are to:

1. Develop new tests of fit for the Weibull distribution with known shape using normalized sample spacings.
2. Generate critical values of the test statistics at various significance levels using complete data.
3. Generate critical values of the test statistics at various significance levels using censored data. (Censoring is explained in Chapter 2, Section 2.3.)
4. Compute the power of the new tests against a broad range of alternative distributions.
5. Compare the power of the new tests with their prominent competitors.

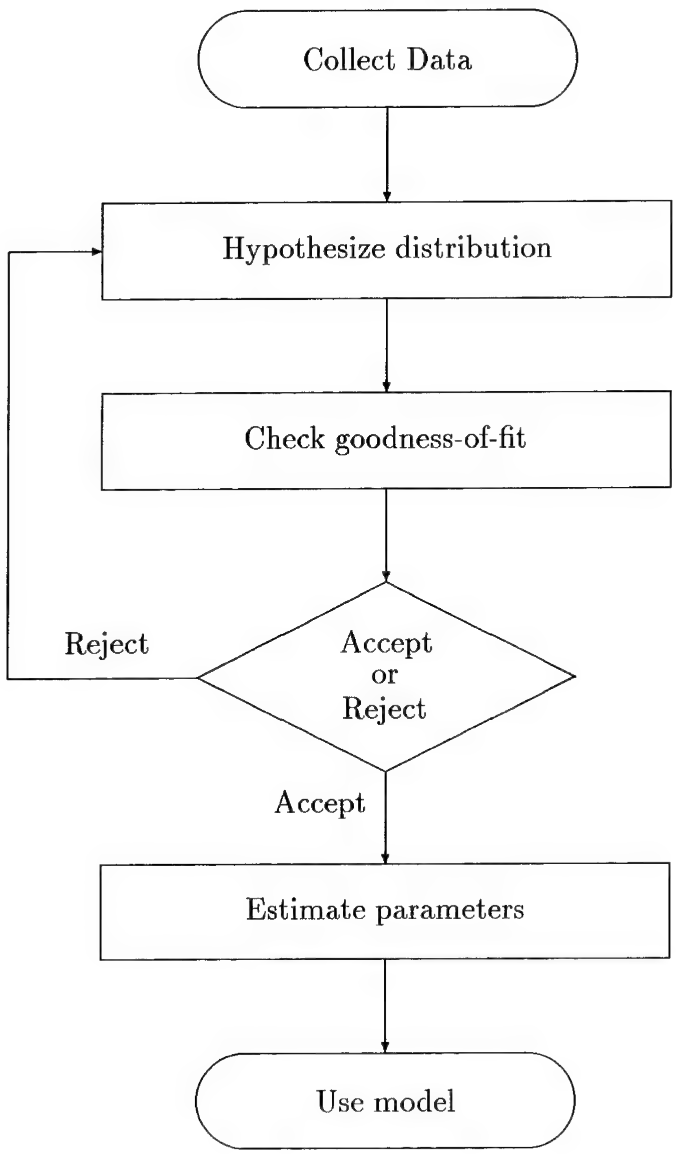

Figure 1.3 Proposed Input Modeling Process

II. LITERATURE REVIEW

2.1 The Three-parameter Weibull Distribution

A random variable X has a Weibull distribution if its probability density function (pdf) and its cumulative density function (cdf) are given by

$$f(x;\delta,\theta,\beta) = \frac{\beta}{\theta}\left(\frac{x-\delta}{\theta}\right)^{\beta-1} \exp\left\{-\left(\frac{x-\delta}{\theta}\right)^{\beta}\right\}$$

$$\text{and} \quad F(x;\delta,\theta,\beta) = 1 - \exp\left\{-\left(\frac{x-\delta}{\theta}\right)^{\beta}\right\}.$$

The three parameters of the Weibull distribution are θ, which is the scale parameter, sometimes called the characteristic life; β, which is the shape parameter; and δ, which is the location parameter or minimum life.

The mean and variance of a Weibull distributed random variable X are given by

$$\mu = \theta\Gamma\left(1+\frac{1}{\beta}\right)$$

$$\text{and} \quad \sigma^2 = \theta^2\left(\Gamma(1+\frac{2}{\beta}) - \Gamma^2(1+\frac{1}{\beta})\right).$$

Here $\Gamma(\cdot)$ is the gamma function defined as

$$\Gamma(\cdot) = \int_0^\infty x^{(\cdot)-1}e^{-x}dx.$$

2.2 History and Application of the Weibull Distribution

The Weibull probability density function was developed by Swedish scientist, Waloddi Weibull, in 1939. In his research, he investigated the underlying distribution of the phenomenon of rupture in solids. In 1951, Weibull published a paper which

demonstrated the use of the distribution in a study of the yield strength and fatigue of steel, the size distribution of fly ash, and the fiber strength of cotton (32:293-297).

Today, the Weibull distribution is a widely used lifetime distribution in the field of reliability. Mann et al. define reliability as the probability of a device performing its defined purpose adequately for a specified period of time, under the operating conditions encountered (24:1). With this in mind, the parameters of the Weibull distribution can be defined in terms of reliability. As shown in Figure 2.1, the location parameter or minimum life, δ, indicates the value of a random variable X for which failures may begin to occur. For example, when $\delta = 0$, failures can occur immediately after a device is put into operation. A device with a minimum life $\delta > 0$ has a period of time that is failure free. In addition to minimum life, it is well known that many components and systems experience three distinct phases during their operation (burn-in, useful life, and wear-out). The shape parameter of the Weibull distribution can be used to describe these phenomena, as illustrated in Figure 2.2. The shape parameter, β, determines the failure rate direction as follows

- When $\beta < 1$, the failure rate decreases with time. Decreasing failure rate is common during the burn-in period of a component or system.
- When $\beta = 1$, the failure rate is constant. Constant failure rate describes the useful life of a component or system.
- When $\beta > 1$, the failure rate increases with time. Increasing failure rate is common during the wear-out period of a component or system.

Finally, the scale parameter or characteristic life, θ, specifies the dispersion of a Weibull random variable X about its mean. Examples of dispersion for a given location and shape parameter are shown in Figure 2.3.

In reliability theory, failure rates are often described with the use of the hazard function. This function is particularly useful since it describes how the instantaneous probability of failure for a device changes with time (18:8-13). In other words,

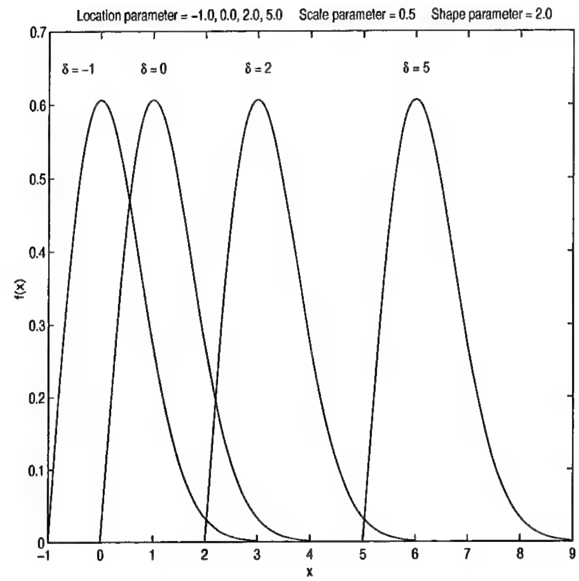

Figure 2.1 Weibull Distributions with Various Location Parameters

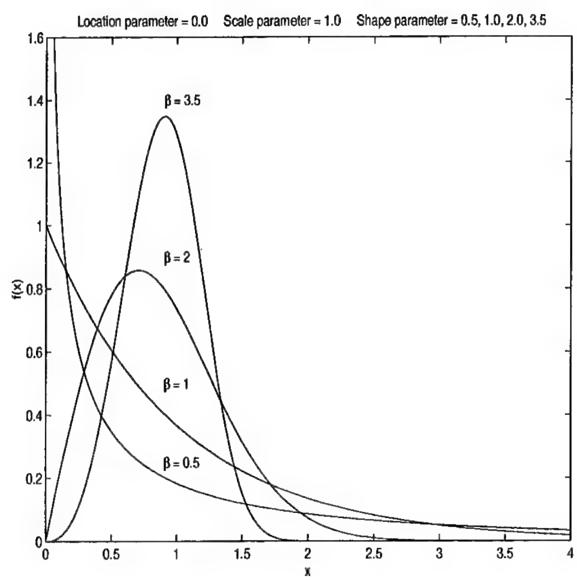

Figure 2.2 Weibull Distributions with Various Shape Parameters

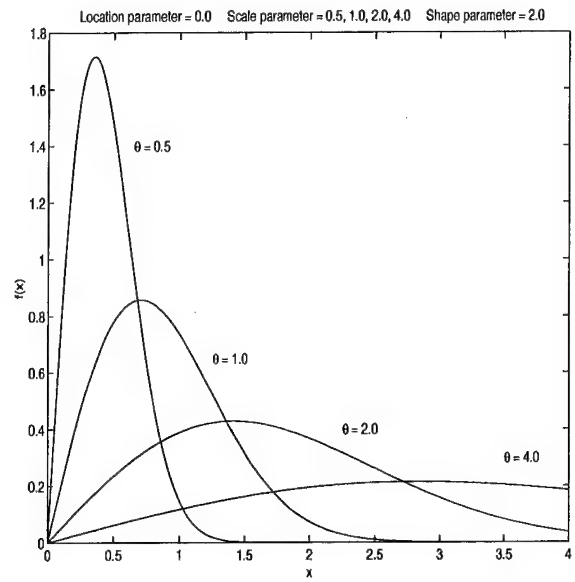

Figure 2.3 Weibull Distributions with Various Scale Parameters

the hazard function is the instantaneous failure rate. Formally, let $F(x)$ be the distribution function of the time-to-failure random variable X, and let $f(x)$ be its probability density function. Then the hazard rate, $h(x)$, is defined as

$$h(x) = \frac{f(x)}{1 - F(x)}.$$

The denominator is called the reliability function, $R(x)$. For a random variable which follows the Weibull failure model, the hazard function has the form

$$h(x) = \frac{\beta}{\theta}\left(\frac{x-\delta}{\theta}\right)^{\beta-1},$$

with the reliability function given by

$$R(x; \theta, \beta, \delta) = \exp\left\{-\left(\frac{x-\delta}{\theta}\right)^{\beta}\right\}.$$

$R(x)$ is easily seen to be the complement of the cumulative distribution function of X.

2.3 Concept and Types of Censoring

Lifetime data analysis is often complicated by the introduction of censored samples. Many times only a fraction of the entities under investigation have known lifetimes. In some cases, it may be justifiable to neglect sampling restrictions. However, if sampling restrictions are severe, valid analysis of the data requires the censoring to be considered (6:v). Fortunately, goodness-of-fit statistics have been adapted for all forms of censoring. The most common and simple censoring schemes involve a planned limit to the magnitude of the variables under study or to the number of observations. These are called Type I and Type II censored data, respectively, sometimes referred to as "time censoring" and "failure censoring" (28:367).

2.3.1 Type I Censoring. When life test experiments are terminated at a predetermined length of time, L, the resulting data is said to be Type I censored. In such situations, an entity's lifetime will only be known exactly if it is less than or equal to the predetermined time period. More precisely, a Type I censored sample arises when items $1, 2, \ldots, n$ are subjected to limited periods of observation L_1,L_2,...,L_n, such that an item's lifetime, T_i is observed only if $T_i \leq L_i$ (18:34-37).

2.3.2 Type II Censoring. When life test experiments are terminated at the time of the rth item failure, the resulting data is said to be Type II censored. Typically, the data consists of the r smallest lifetimes $T_{(1)} \leq T_{(2)} \leq \ldots \leq T_{(r)}$ out of a random sample of n lifetimes $T_1, \ldots, T_n$ from the statistical failure model being studied. Tests involving Type II censoring are common in practice because they can save time and money, since in some cases it could take a long time for all items to fail (18:32-34).

If we define L to be a particular lifetime value, then an observation is said to be right censored at L if its value is greater than or equal to L and unknown. Similarly, an observation is said to be left censored at L if it is only known to be less than or equal to L (18:31). Situations can also arise in which the data is both

left and right censored (28:462). This thesis addresses left, right, and combinations of left and right Type II censored data.

2.4 Classical Goodness-of-fit Tests

In general, goodness-of-fit tests are used to determine if a sample data set could have come from a population described by a commonly known distribution function. This is accomplished by measuring how well the hypothesized distribution fits the data. Such procedures are discussed extensively by Stephens (28). Although graphical goodness-of-fit techniques can be employed, more formal methods using test statistics are most abundant in the literature. The main categories of goodness-of-fit tests are chi-squared tests, empirical distribution function tests, and smooth tests. For any given test type, goodness-of-fit can also be distinguished as "omnibus tests" and "directional tests". Omnibus tests are designed to be effective against a broad class of alternatives to a given hypothesized distribution, while directional tests are designed to effectively detect specific types of departure from the hypothesized distribution, such as symmetry or skewness (18:431).

2.4.1 Chi-squared Type Tests. In 1900, Karl Pearson introduced a system of distributions to provide alternative models to the normal distribution assumption of biological populations. Consequently, the need to test the fit of his proposed models resulted in the chi-squared test statistic (28:63). The chi-squared test formalizes the intuitive idea of comparing a histogram of the sample data to the shape of the proposed probability density or mass function (4:375). The test requires that data be placed in class intervals. The number of classes and the interval widths directly impact the values of the calculated and tabulated chi-square statistic. Since the data must be grouped into cells, some information about the underlying distribution is lost, often making chi-squared type tests less powerful than others (28:63). For example, when the data is grouped one way, the null hypothesis may be accepted, but when grouped another way, the hypothesis may be rejected. This phenomenon

is most apparent when the data is hypothesized to have come from a continuous distribution, such as a lifetime distribution, since the grouping of the data is arbitrary. Hence, chi-squared tests of fit are most generally applicable when the sample data is hypothesized to have come from a discrete distribution. In the case of continuous random variables, such as time, tests based on empirical distribution function statistics generally yield much higher power (28:110).

2.4.2 Empirical Distribution Function Type Tests. The empirical distribution function, or *EDF*, is a step function which estimates the population distribution of the sample from which it is calculated. The EDF of n ordered data points $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is commonly given by

$$F_n(x) = \begin{cases} 0, & x < X_{(1)} \\ \frac{i}{n}, & X_{(i)} \leq x < X_{(i+1)}, \qquad i = 1, \ldots, n-1 \\ 1, & x \geq X_{(n)} \end{cases}$$

Divided into two classes, the supremum class and the quadratic class, EDF statistics measure the vertical distance between the EDF, $F_n(x)$, and a given distribution function, $F(x)$, to determine the adequacy of the fit (28:97). Figure 2.4 shows an example of how $F_n(x)$ and $F(x)$ might compare. Three test statistics widely used are Kolmogorov-Smirnov (K-S), Cramér-von Mises (C-vM), and Anderson-Darling (A-D). When the given distribution function is completely specified and data are uncensored, tests based on these statistics are distribution-free and their percentage points all generally known (18:432-433). In other words, the percentage points are the same no matter which theoretical distribution is given in the null hypothesis of the statistical test. However, when parameters are estimated from the sample data, EDF tests require critical values which depend upon the hypothesized distribution, the sample size, the parameters estimated, and the method of estimation, as discussed later in this chapter.

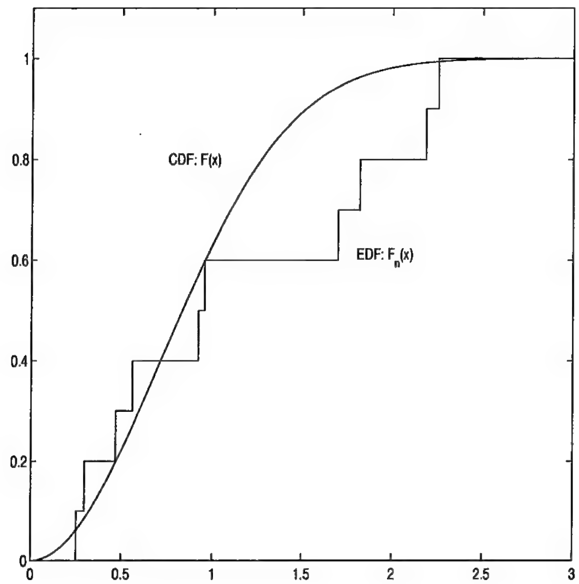

Figure 2.4 Comparison of an EDF to a Hypothesized CDF

2.4.2.1 Kolmogorov-Smirnov Test Statistics. The K-S test statistic is based on the following two supremum statistics

$$D^+ = \max[(i/n) - Z_i] \tag{2.1}$$

and

$$D^- = \max[Z_i - (i-1)/n], \tag{2.2}$$

where $Z_i = F(x_i)$ for $i = 1, 2, \ldots, n$; known as the probability integral transform. Hence, D^+ is the largest vertical difference when $F_n(x)$ is greater than $F(x)$ and D^- is the largest vertical difference when $F_n(x)$ is smaller than $F(x)$. The K-S statistic is computed from Equations 2.1 and 2.2 as

$$D = \max[D^+, D^-].$$

Although the K-S statistic is the most well-known EDF statistic, it is often much less powerful than the quadratic statistics, such as the Cramér-von Mises and the Anderson-Darling statistics (28:110).

2.4.2.2 Cramér-von Mises Test Statistic. The C-vM test statistic, denoted by W^2, is a particular member of a family of quadratic statistics of the form

$$Q = n \int_{-\infty}^{\infty} [F_n(x) - F(x)]^2 \psi(x) dF(x), \tag{2.3}$$

where $\psi(\cdot)$ is a weight function which gives weights to the squared difference $[F_n(x) - F(x)]^2$. For the C-vM test statistic, W^2, $\psi(x) = 1$ for all x. Stephens (27) gives the computational form of the C-vM statistic as

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(Z_i - \frac{2i-1}{2n} \right)^2, \tag{2.4}$$

where $Z_i = F(x_i)$ for $i = 1, 2, \ldots, n$.

2.4.2.3 Anderson-Darling Test Statistic. The A-D test statistic is a special case of Equation 2.3 with weight function $\psi(x) = [F(x)[1 - F(x)]]^{-1}$. This weight function gives more emphasis to differences in the tails of the distribution. As a result, A^2 behaves similarly to W^2, but is generally more powerful for tests when $F(x)$ departs from the true distribution function in the tails (28:110). The A-D test statistic, denoted by A^2, is given by

$$A^2 = n \int_{-\infty}^{\infty} [F_n(x) - F(x)]^2 [F(x)[1 - F(x)]]^{-1} dF(x). \tag{2.5}$$

Stephens' (27) computational formula is

$$A^2 = -n - \frac{1}{n} \sum_{i=1}^{n} (2i - 1)[\log(Z_i) + \log(1 - Z_{n+1-i})], \tag{2.6}$$

where $Z_i = F(x_i)$ for $i = 1, 2, \ldots, n$. When it is important to detect departures in the tails of distributions, the A-D statistic is the recommended statistic (28:110).

2.5 Parameter Estimation

Occasionally, goodness-of-fit tests are used to test hypotheses of distributions with all parameters known (fully specified), but usually the hypothesized distribution will involve unknown parameters (18:431). Therefore, the parameters typically must be estimated from the observed data prior to initiating any tests. Methods of parameter estimation include, but are not limited to, maximum-likelihood estimation (MLE), minimum-distance estimation (MDE), and methods based on best linear unbiased estimators (BLIEs), best linear invariant estimators (BLUEs), and good linear unbiased estimators (GLUEs).

The most common and accepted method of parameter estimation is the method of maximum-likelihood because it tends to yield estimators with good properties. For instance, this method often leads to minimum-variance unbiased estimators (MVUEs). A particularly attractive property is the invariance property of MLEs. In general, if the cumulative distribution function of an EDF statistic is not fully specified, the distribution of the EDF statistic will depend on the distribution being tested, the sample size n, the parameters estimated, and the method of estimation (28:102). However, in a paper published in 1948, David and Johnson showed that the distribution of any EDF statistic is simplified when the unknown parameters are the location and scale. They demonstrated that when the location and scale parameters of the hypothesized distribution function are estimated from the data using invariant estimators, the distribution of any EDF statistic will depend only on the functional form of the distribution being tested and will not depend on the true values of the unknown parameters (10:182-190). An estimator is invariant when sample data transformed by $aX_i + b$ is equal to $a\hat{\theta} + b$, where $\hat{\theta}$ is the original estimate value. This property allows the estimation of one set of location and scale parameters to generalize any set of location and scale parameters.

2.5.1 Maximum-Likelihood Estimation (MLE). The concept of MLE is to determine the estimate which maximizes the probability of obtaining the observed sample from a given distribution. In other words, this technique selects as estimates the values of the parameters that maximize the likelihood (joint probability function or joint density function) of the observed sample (31:399). The likelihood function for the Weibull distribution is given by

$$\begin{aligned} L &= f(x_1, x_2, \ldots, x_n; \theta, \beta, \delta) = \prod_{i=1}^{n} f(x_i; \theta, \beta, \delta) \\ &= \beta^n \theta^{-\beta n} \prod_{i=1}^{n} (x_i - \delta)^{\beta - 1} \exp\left\{ -\theta^{-\beta} \sum_{i=1}^{n} (x_i - \delta)^{\beta} \right\}. \end{aligned}$$

The maximum is found by taking the partial derivatives of Equation 2.7 with respect to each parameter and setting the resulting equations equal to zero. Fortunately, since the natural logarithm of L is a monotonically increasing function of L, both L and $\log(L)$ are maximized by the same parameter values (31:400). Taking the natural logarithm of the likelihood function simplifies taking the derivative since the product of density functions is converted into a summation as follows

$$\log(L) = n\log(\beta) - n\beta\log(\theta) + (\beta - 1)\sum_{i=1}^{n}\log(x_i - \delta) - \theta^{-\beta}\sum_{i=1}^{n}(x_i - \delta)^{\beta}. \quad (2.7)$$

The partial derivatives of equation 2.7 with respect to the scale, location, and shape parameters (θ, β, and δ, respectively) are given by

$$\frac{\partial \log(L)}{\partial \theta} = -\frac{n\beta}{\theta} + \beta\theta^{-(\beta+1)}\sum_{i=1}^{n}(x_i - \delta)^{\beta},$$

$$\frac{\partial \log(L)}{\partial \beta} = \frac{n}{\beta} - n\log(\theta) + \sum_{i=1}^{n}\log(x_i - \delta) - \theta^{-\beta}\sum_{i=1}^{n}(x_i - \delta)^{\beta}\log(x_i - \delta),$$

$$\text{and} \quad \frac{\partial \log(L)}{\partial \delta} = (1 - \beta)\sum_{i=1}^{n}(x_i - \delta)^{-1} + \beta\theta^{-\beta}\sum_{i=1}^{n}(x_i - \delta)^{\beta - 1}.$$

Since these equations are impossible to solve explicitly for the three distribution parameters, an iterative numerical approach, such as the procedure developed by Harter and Moore (14:639-643) must be employed.

2.6 *Modified EDF Statistics*

As mentioned previously, when a given distribution function in the null hypothesis of a statistical test is completely specified and the sample data is uncensored, tests based on the EDF statistics do not depend on the distribution being tested. Under these conditions, tables of percentage points (critical values) for each type of test, such as those presented by Stephens (28), determine the size of the test's rejection region at various significance levels. Goodness-of-fit tests used to test hypotheses of distributions with parameters estimated from the sample data are said to be "modified". Modified EDF tests require critical values which depend upon the hypothesized distribution function, the parameters estimated, the sample size, and the method of estimation, but not upon the values of the true estimated parameters (28:102). Critical values of this type can usually be obtained only through Monte Carlo simulation (33:154), as discussed in the next chapter.

2.7 *Order Statistics and Spacings*

Many functions of random variables depend on the relative magnitudes of a sample of observations. When random variables are ordered according to their magnitudes, the resulting ordered variables are called order statistics (31:278). Formally, suppose a random variable X has a continuous probability density function $f(x)$ and distribution function $F(x)$ and $X_1, X_2, \ldots, X_n$ is a random sample from this distribution. The sample observations rearranged in order of magnitude, denoted by

$$X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)},$$

are called the order statistics of the sample. The distances between successive order statistics are often referred to as gaps or spacings. Order statistics are particularly important in life-testing analyses since they may often be observed directly from the experiment (3:46). For example, suppose a random selection of n cockpit warning lights are placed on life test. If the j^{th} warning light is the first to fail, X_j would be recorded as $X_{(1)}$. Similarly, the second failure time would be recorded as $X_{(2)}$, and so on, without regard to which warning lights actually produced failures. Such a sample is called an ordered sample and the spacings are defined as

$$S_i = X_{(i+1)} - X_{(i)}, \quad i = 1, \ldots, n-1$$

where $X_{(i)}, X_{(i+1)}, \ldots, X_{(n)}$ are the n order statistics of the random sample above. Suppose m_i is the expected value of the ith order statistic of a random sample from $F(x)$, then the normalized spacings are given by

$$Y_i = \frac{S_i}{m_{i+1} - m_i}, \quad i = 1, \ldots, n-1.$$

Normalized spacings provide useful tests of fit for many continuous distributions and can be used with samples which are Type II censored at either or both ends (21). For censored data, it is convenient to define the normalized spacings as

$$Y_i = \frac{X_{(k+i)} - X_{(k+i-1)}}{m_{k+i} - m_{k+i-1}}, \quad i = 1, \ldots, r-1$$

where r is the censored sample size and k is the index of the first order statistic of this censored sample. Pyke (25) shows that the normalized spacings of any continuous distribution are asymptotically exponentially distributed, and are asymptotically independent. Once normalized spacings are found, they can be transformed to produce a set of $\tilde{Z}_i$ values (between 0 and 1) which can be used in place of the Z_i values in the computation of EDF test statistics described earlier.

2.8 *Tests for the Weibull Distribution*

The majority of the literature deals with goodness-of-fit tests for the two-parameter form of the Weibull distribution in which the location parameter δ is assumed to be zero. When this assumption can not be made safely, the location parameter is often estimated by the smallest order statistic of the sample data under test and is then transformed to the origin, yielding a "known" parameter of the distribution. Consequently, some data, and thus information about the underlying distribution, is lost. With location parameter $\delta = 0$, tests for the two-parameter Weibull are equivalent to tests for the extreme-value distribution if the raw data is log-transformed by

$$Y_i = \log X_i \quad i = 1, \ldots, n$$

where $X_1, X_2, \ldots, X_n$ are the raw sample observations. As discussed in Chapter 1, though, there are various applications of the Weibull in which the shape parameter of the distribution is known, while the location and scale parameters may be unknown. Simply transforming the data as described above and testing it to be from the extreme-value distribution may yield misleading results. Fortunately, goodness-of-fit tests for the Weibull distribution with known shape are available. With all this in mind, the following two sections highlight the goodness-of-fit tests for the Weibull distribution that are most applicable to the goals of this thesis. Many other tests have been constructed as well, but these are of less direct relevance to this research and are not presented here.

2.8.1 Test Based on Modified EDF Statistics. In 1981, Bush (5) generated the modified Anderson-Darling and Cramér-von Mises rejection tables for the Weibull distribution with known shape using invariant estimates of the location and scale parameters. Bush also conducted an extensive power study which compared the powers of his modified A^2 and W^2 tests to the modified $K-S$ test for the Weibull with known shape, developed by Cortes (8), and the χ^2 goodness-of-fit test. The

power study shows that the modified Anderson-Darling and Cramér-von Mises tests exhibit the best power. The superiority of tests based on the A^2 and W^2 statistics are also noted by Stephens (28).

2.8.2 Tests Based on Normalized Spacings. The properties of order statistics and their spacings make them useful in tests of fit. Hence, several goodness-of-fit tests based on order statistics are present in the literature for the Weibull distribution.

Mann, Scheuer, and Fertig (1973) develop the S statistic for tests of fit of data to the two-parameter Weibull distribution ($\delta = 0$) and, equivalently, the extreme-value distribution. Their test, specifically designed to work with censored samples, exploits the fact that the right-hand tail of the extreme value density function is "shorter" than that of usual appropriate alternative distributions, while the left-hand tail is "longer". The S statistic is given by

$$S = \frac{\sum_{i=\frac{n}{2}+1}^{n-1} G_i}{\sum_{i=1}^{n-1} G_i},$$

where G_i, defined by

$$G_i = \frac{X_{(i+1)} - X_{(i)}}{m_{i+1:n} - m_{i:n}},$$

is the ratio of the difference of successive order statistics to the difference of their expected values. As previously mentioned, spacings of this type are said to be normalized since the spacings between ordered observations are standardized by dividing by known constants. The S statistic is simpler to calculate than most classical tests of fit since the scale and shape parameters of this particular Weibull family do not need to be known nor estimated. Mann et al. (22) provide tables of percentage points and expected value differences needed to implement this statistic. Also included in their study is a power comparison which reveals the S statistic is

at least as good as its competitors, namely the two-sided Kolmogorov-Smirnov test, the Cramér-von Mises test, the Anderson-Darling test, and the Kuiper test (22).

Littell, McClave, and Offen (1979) conduct an extensive power study involving Mann, Scheuer, and Fertig's S statistic along with the prominent EDF statistics for the two-parameter Weibull distribution, when the parameters are estimated. Their results show the S statistic does well relative to the others with the exception of two of the six alternative distributions in their study (19).

Tiku and Singh (1981) extend the pioneering work of Mann, Scheuer, and Fertig by applying Tiku's Z^* and Z goodness-of-fit statistics to the two-parameter Weibull distribution and, equivalently, the extreme-value distribution. Tiku (29) defines the Z^* statistic for a Type II censored sample as

$$Z^* = \frac{2\sum_{i=r+1}^{n-s-2}(n-s-1-i)G_i}{(m-2)\sum_{i=r+1}^{n-s-1} G_i},$$

where, similar to Equation (24), G_i is given by

$$G_i = \frac{Y_{(i+1)} - Y_{(i)}}{m_{i+1:n} - m_{i:n}},$$

in which $Y_i = \log X_i$, for $i = r+1, \ldots, n-s$ and $m_{i:n}$ is the expected value of the rth order statistic. The censored sample $x_{r+1}, x_{r+2}, \ldots, x_{n-s}$ is obtained by rearranging a random sample of size n in ascending order of magnitude and censoring the $r(\geq 0)$ smallest and the $s(\geq 0)$ largest observations, where $m = n - r - s$. The Z^* statistic is location and scale invariant and is available for complete as well as censored data. The Z statistic is of the same form as Z^*, but with G_i replaced by the spacings $D_i = (n-i)(U_{i+1} - U_i)$, where $U_i = X_i^{\alpha^*}$. In this notation, α^* is an estimate of the scale parameter. Since it is difficult to estimate α from censored samples, the use of the Z statistic in tests of fit for the Weibull is recommended for complete samples ($r = s = 0$) only (30). Much like, Littell et al. (1979), Tiku and Singh perform power comparisons which reveals that both Z^* and Z are, on the whole,

more powerful than their prominent competitors for complete samples, as well as censored samples.

Lockhart, O'Reilly, and Stephens (1986) also study tests for the extreme value and two-parameter Weibull distributions based on normalized spacings from complete and censored samples. In addition to the statistics proposed by Mann et al. (22) and Tiku and Singh (30), Lockhart et al. study the Anderson-Darling spacings test given by

$$A^{*2} = -(n-2) - (n-2)^{-1} \sum_{i=1}^{n-2} (2i-1)\{\log Z_i + \log(1 - Z_{n-i-1})\},$$

where Z_i are computed from normalized spacings of the form

$$Y_i = \frac{X_{(i+1)} - X_{(i)}}{m_{i+1:n} - m_{i:n}}, \qquad i = 1, 2, \ldots, n-1,$$

in which m_i is the expected value of the ith order statistic of a random sample of size n drawn from the hypothesized distribution. Lockhart et al. point out that the tests suggested by Mann et al. (22) and Tiku and Singh (30) are tests based, respectively, on the median and mean $\bar{Z}$ of the z_i. Also, Tiku and Singh's (30) statistic Z^* can be shown to be $2\bar{Z}$ (20). For the alternatives considered in their power comparisons of the A^{*2}, $\bar{Z}$, and S statistics, the A^{*2} and $\bar{Z}$ show comparably better power than the S statistic. However, Lockhart et al. recommend A^{*2} overall, due to the possible inconsistency of the results based on $\bar{Z}$ (20:413-421).

In 1993, Coppa (7) implemented a derivative of Tiku and Singh's Z^* statistic and developed a test for the Weibull distribution with known shape in which the raw data of full samples (no censoring) is used. This statistic, also denoted by Z^* is given by

$$Z^* = \frac{2\sum_{i=r+1}^{n-2}(n-1-i)G_i}{(n-2)\sum_{i=1}^{n-1} G_i},$$

where, again, G_i is the ratio of the difference of successive order statistics to the difference of their expected values. given by

$$G_i = \frac{X_{(i+1)} - X_{(i)}}{m_{i+1:n} - m_{i:n}},$$

for $i = 1, \ldots, n$. Percentage points for this test are presented for a variety of sample sizes, shape parameters, and significance levels. A power study is also performed which reveals that Z^* is a directional test; it performs extremely well when the null hypothesis is highly skewed, but performs poorly when the null is more symmetric. Duman (11) also discovered this phenomenon in his implementation of Z^* for testing the gamma distribution.

2.9 Summary

The wide variety of geometries assumed by the Weibull distribution allow it to generalize many sets of data. This flexibility has made the Weibull distribution popular in reliability studies. Clearly though, it is not restricted to such studies and is suitable for a broad range of applications. However, the appropriateness of the Weibull distribution to model a particular set of data depends on how well it conforms to the data. Hence, goodness-of-fit tests are used to assess the validity of model and data correspondence.

There are several goodness-of-fit tests available for the Weibull distribution in the literature which exhibit good power. Some of these procedures require estimation of the distribution parameters and some do not. Since tests based on sample spacings are among those tests which do not require parameter estimation, their test statistics are easier to compute. Thus, tests of this type are more desirable if good power can be achieved against a broad range of alternatives. Consequently, tests based on spacings are the focus of this research effort. Specifically, the Anderson-Darling spacings test, recommended by Lockhart et al. (20), as well as the Cramér-von Mises spacings test,

will be developed for the Weibull distribution with known shape parameter. Both full samples and Type II censored samples will be investigated.

III. METHODOLOGY

3.1 Introduction

The goal of this thesis is to develop more powerful omnibus goodness-of-fit tests for the three-parameter Weibull distribution with the benefit of eliminating the need for location and scale estimation (prior to initiating the tests) when the shape parameter is known. This will be accomplished by using a Monte Carlo method for obtaining the critical values of the Anderson-Darling and Cramér-von Mises test statistics based on normalized spacings. This chapter discusses the procedures used to obtain the critical values as well as the those employed in determining the power of the tests.

3.2 The Test Statistics

The test statistics used throughout this thesis are given in computational form as

$$A_s^2 = -r - \frac{1}{r}\sum_{i=1}^{r}(2i-1)[\log(Z_{(i)}) + \log(1 - Z_{(r+1-i)})] \tag{3.1}$$

by Lockhart et al. (21), and

$$W_s^2 = \frac{1}{12r} + \sum_{i=1}^{r}\left(Z_{(i)} - \frac{2i-1}{2r}\right)^2. \tag{3.2}$$

These expressions are identical to the ones given by Equations 2.4 and 2.6 introduced in Chapter 2. However, here r is the number of $Z_{(i)}$ transformations of the normalized spacings, calculated as follows:

1. Let $X_{(k)}, X_{(k+1)}, \ldots, X_{(k+r+1)}$ be the $r+2$ order statistics of a random sample of size n which is left and right Type II censored.

2. Calculate $r+1$ normalized spacings

$$Y_i = \frac{X_{(k+i)} - X_{(k+i-1)}}{m_{k+i} - m_{k+i-1}}, \quad i = 1, \ldots, r+1$$

where m_i is the expected value of the ith order statistic of a random sample from

$$F(x; \theta, \beta, \delta) = 1 - \exp\left\{ -\left(\frac{x-\delta}{\theta} \right)^{\beta} \right\}$$

with $\delta = 0$ and $\theta = 1$, given by

$$E(X_{m,n}; \theta, \beta) = \theta n \binom{n-1}{m-1} \Gamma\left(1 + \frac{1}{\beta}\right) \sum_{j=0}^{m-1} \binom{m-1}{j} \frac{-1^{(m+j-1)}}{(n-j)^{(1+\frac{1}{\beta})}} \tag{3.3}$$

However, implementation of this equation in Matlab results in problems with computer accuracy, as discussed in Chapter 4. Fortunately, Harter (13) provides tables of these values for the Weibull distribution for various shape parameters and sample sizes. In an effort to curtail computer accuracy problems and to increase the efficiency of the code, Harter's values are used in this research effort via a table look-up procedure.

3. Let T_j be the partial sum $\sum_{i=1}^{j} Y_i$ and compute

$$Z_{(i)} = \frac{T_i}{T_{r+1}}, \quad i = 1, \ldots, r.$$

As a result, the $Z_{(i)}$ are in ascending order between 0 and 1. A numerical example of the computation of A_s^2 and W_s^2 from a right-censored sample is provided in the next section.

3.2.1 A Numerical Example. Suppose in a reliability study of 10 items on test, we record only the failure times of the first 8 items. In other words, the sample is censored at the 8th observation. This is an example of right-Type II censoring. As a result, our right censoring level is 0.8 (8 out of 10 potential observations), while

our left censoring level is zero. Let the ordered observations (failure times) from our test be:

20.1 22.3 24.4 24.5 24.8 26.0 27.6 28.2.

Suppose further that we wish to test the null distribution that these observations come from a Weibull distribution with shape parameter, $\beta = 1$. To calculate either the A_s^2 or W_s^2 test statistics, we need the expected values $m_{r:n}$ of a Weibull($\beta = 1$) sample of size 10. These are obtained either from Equation 3.3 or from Harter (13), and for $n = 10$ and $m = 1,2,\ldots,8$, are given by

0.1000 0.2111 0.3361 0.4790 0.6456 0.8456 1.0956 1.4290.

From these eight expected values and eight observations, seven normalized spacings will result giving rise to seven T values. The normalized spacings are found to be

19.1719 17.0400 0.8397 1.7407 6.0500 6.2400 1.7696,

yielding T-values of

19.1719 36.2119 37.0517 38.7924 44.8424 51.0824 52.8520.

From these seven T-values, six Z-values will arise. They are computed, and are given by

0.3627 0.6852 0.7010 0.7340 0.8485 0.9665.

Now that we have the $Z_{(i)}$ ($r = 6$) values, we can compute either the A_s^2 or W_s^2 statistics from Equation 3.1 or Equation 3.2. For this example, $A_s^2 = 1.9244$ and $W_s^2 = 0.4021$.

3.3 Monte Carlo Simulation

The distributions of the above test statistics are pivotal to the goodness-of-fit tests on which they are based because the critical values used in the tests are derived

from these distributions. For this thesis, the distributions of the test statistics under the null hypothesis were simulated from numerous random Weibull samples. This type of simulation is commonly known as Monte Carlo simulation. In Monte Carlo simulations, random deviates are used to solve certain mathematically intractable problems where the passage of time plays no substantive role (17:113-114). In this thesis, each simulation is started with a unique seed for generating random numbers. Use of the Monte Carlo method will be illustrated with greater detail in subsequent discussions.

3.4 Computation of the Critical Values

The Monte Carlo procedure for obtaining the critical values of the A_s^2 and W_s^2 statistics is detailed in the following steps:

1. *Generate random deviates.* For a given sample size n and Weibull shape parameter β, random deviates are generated from the Weibull distribution using the WEIBRND function in MATLAB with scale parameter $\theta = 1$ and location parameter $\delta = 0$. As mentioned in Chapter 2, the critical values for the Weibull distribution with known shape are independent of these parameters.

2. *Order the sample.* The n Weibull random deviates are sorted in ascending order. This is necessary for the spacings calculations and for generating a censored sample.

3. *Censor the sample.* For a given combination of left and right Type II censoring, a "censored sample" is generated by discarding the appropriate portion of the ordered sample. For full samples, this step is omitted.

4. *Compute the test statistics.* The procedure described at the beginning of this chapter is used to determine the values of the modified Cramér-von Mises and Anderson-Darling test statistics.

5. *Repeat steps 1 through 4.* Steps 1-4 are repeated 100,000 times, thus generating 100,000 independent A_s^2 and W_s^2 values. A vector for each statistic is used in MATLAB to store these values for further computations.

6. *Order the test statistics.* The 100,000 A_s^2 and W_s^2 test statistics are sorted in ascending order. This is necessary to calculate percentage points.

7. *Determine the critical values.* The 75th, 80th, 85th, 90th, 95th, 97.5th, and 99th percentage points are computed from each test statistic vector using the PRCTILE function in MATLAB. These percentiles comprise the critical values for the test statistics.

Figure 3.1 summarizes the critical value generation process in flowchart form. This process is repeated for sample sizes of 5(5)40 and for shape parameters $\beta = 0.5(0.5)4.0$. The censoring levels employed for each sample size are shown in Table 3.1.

Table 3.1 Type II Censoring Levels

Left censoring	Right censoring
0.00	1.00
0.00	0.80
0.00	0.60
0.20	1.00
0.20	0.80
0.40	1.00

3.5 Power Study

The power of a goodness-of-fit test is defined to be the probability of rejecting the null hypothesis when it is false. It is common to survey and compare the powers of several tests against a variety of statistical distributions to determine the usefulness of a given test. In this thesis, the powers of the Anderson-Darling and Cramér-von Mises tests based on normalized spacings are compared with their prominent competitors.

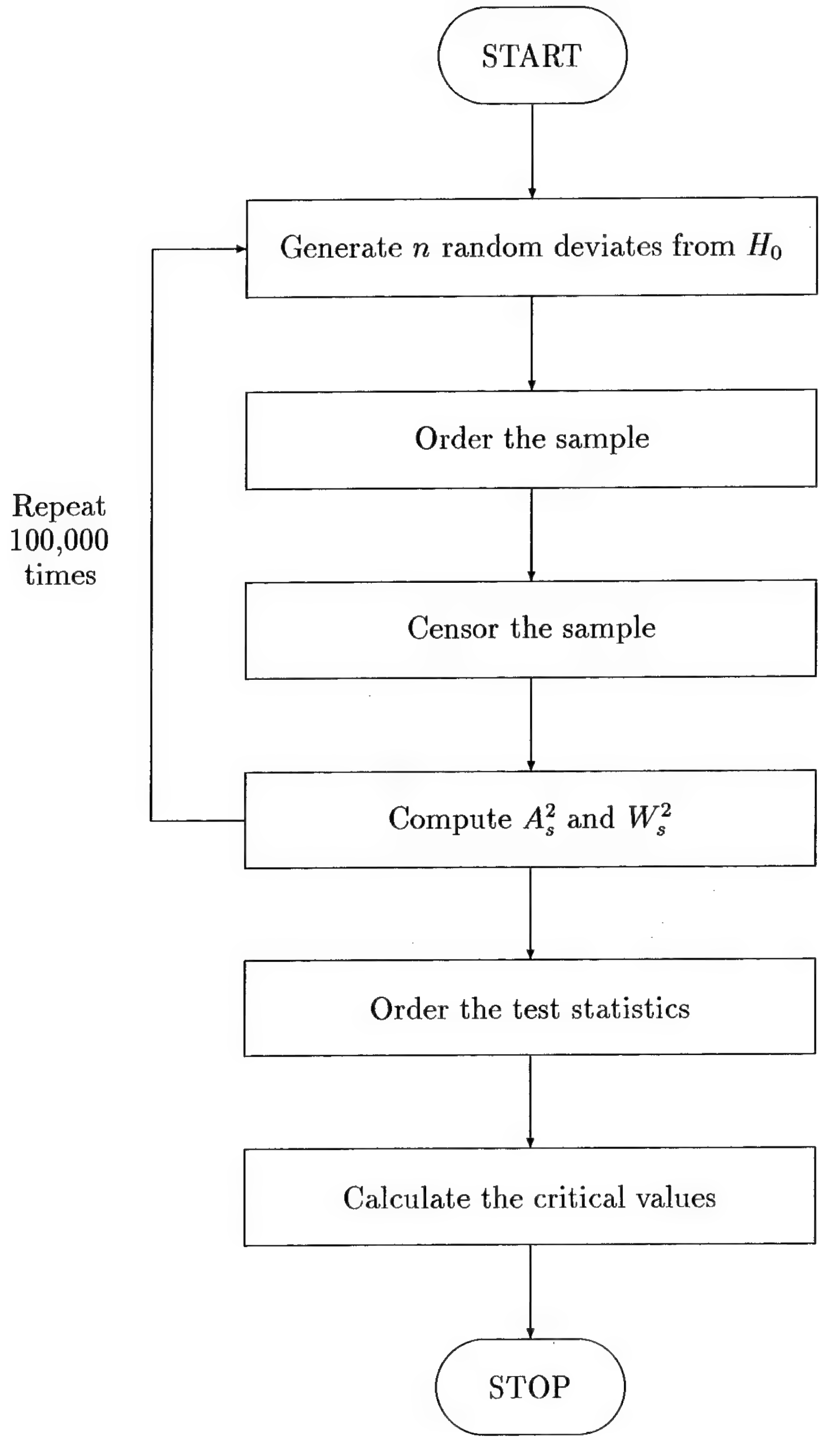

Figure 3.1 Generation of A_s^2 and W_s^2 Critical Values

3.5.1 The Distributions H_0 and H_a. The power of the tests developed in this thesis concern the hypotheses

H_0: a random sample of n X-values follows a Weibull distribution with known shape parameter β, versus

H_a: the random sample of n X-values is not Weibull with shape β.

The selections of H_a used in this study are motivated by previous power studies of goodness-of-fit tests found in the literature. The following statistical families are often used to evaluate the power of goodness-of-fit tests for the Weibull distribution:

1. Weibull:

$$f(x;\theta,\beta,\delta) = \frac{\beta}{\theta}\left(\frac{x-\delta}{\theta}\right)^{\beta-1} \exp\left\{-\left(\frac{x-\delta}{\theta}\right)^{\beta}\right\}, \quad x > \delta;\ \ \theta > 0;\ \ \beta > 0$$

2. Gamma:

$$f(x;\theta,\beta,\delta) = \frac{1}{\theta\Gamma(\beta)}\left(\frac{x-\delta}{\theta}\right)^{\beta-1} \exp\left\{-\left(\frac{x-\delta}{\theta}\right)\right\}, \quad x > \delta;\ \ \theta > 0;\ \ \beta > 0$$

3. Normal:

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^{2}\right\}, \quad -\infty < x < \infty;\ \ -\infty < \mu < \infty;\ \ \sigma > 0$$

4. Uniform:

$$f(x;\delta_1,\delta_2) = \frac{1}{\delta_2-\delta_1}, \quad \delta_1 \le x \le \delta_2$$

5. Beta:

$$f(x;\alpha,\beta) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1;\ \ \beta > 0$$

6. Chi-square:

$$f(x;\nu) = \frac{x^{(\nu-2)/2}e^{-x/2}}{2^{\frac{\nu}{2}}\Gamma(\nu/2)}, \quad x > 0; \quad \nu \in N$$

7. Log-normal:

$$f(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right\}, \quad x > 0; \quad \sigma > 0$$

8. Log-logistic:

$$f(x;\delta,\beta) = \frac{1}{x\beta}\left[\frac{\exp\left\{-\left(\frac{\log x - \delta}{\beta}\right)\right\}}{\left(1+\exp\left\{-\left(\frac{\log x - \delta}{\beta}\right)\right\}\right)^2}\right], \quad x > 0; \quad \beta > 0$$

9. Log-double exponential:

$$f(x;\mu,\delta) = \frac{1}{2x\mu}\exp\left\{-\left(\frac{|\log x - \delta|}{\mu}\right)\right\}, \quad x > 0; \quad \mu > 0$$

10. Log-Cauchy:

$$f(x;\delta,\theta) = \frac{1}{x\pi}\left[\frac{\theta}{\theta^2 + (\log x - \delta)^2}\right], \quad x > 0; \quad \theta > 0$$

In this power study, specific distributions from the above families are selected in an effort to include a broad range of possible alternatives, as well as to compare the results of this thesis to similar research efforts. For these reasons, the following distributions are investigated:

1. Weibull, $\beta = 1.0$ (See Figure 3.2)
2. Weibull, $\beta = 2.0$ (See Figure 3.3)
3. Weibull, $\beta = 3.5$ (See Figure 3.4)
4. Gamma, $\beta = 2.0$ (See Figure 3.5)

5. Normal(0,1) (See Figure 3.6)
6. Uniform(0,1) (See Figure 3.7)
7. Beta(2,2) (See Figure 3.8)
8. Beta(2,3) (See Figure 3.9)
9. Chi-square(1) (See Figure 3.10)
10. Chi-square(4) (See Figure 3.11)
11. Log-normal(0,1) (See Figure 3.12)
12. Log-logistic(0,1) (See Figure 3.13)
13. Log-double exponential(0,1) (See Figure 3.15)
14. Log-Cauchy(0,1) (See Figure 3.14)

These alternative distributions are tested against three different Weibull null distributions (β = 1.0, 2.0, and 3.5) at the 0.1, 0.05, and 0.01 significance levels using complete samples of size 5, 15, and 25. Censored samples with the censoring levels given in Table 3.1 are also considered.

The first eight alternatives listed above allow some direct comparisons with the powers of tests for the Weibull distribution with known shape found in the literature. As shown in the next chapter, the prominent competitors of the tests developed in this thesis are those developed by Bush (5) and Coppa (7). The last six alternatives are chosen to benchmark the power of the tests in this thesis using distributions often employed in power studies of the two-parameter Weibull (location parameter, $\delta = 0$), such as those found in Wozniak (33) and Lockhart et al. (20). Because the logistic, double exponential, Cauchy, and normal distributions are defined on the entire real line, but the Weibull distribution is defined only on the positive half, the random variates of these distributions are transformed to the positive real line. These is accomplished by the transformation $Y_i = \exp(X_i)$.

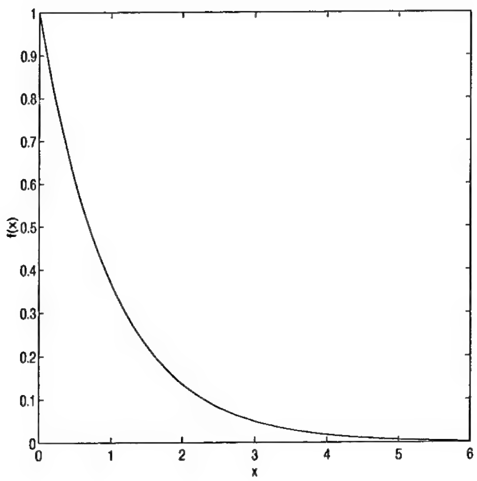

Figure 3.2 Weibull(0,1,1) PDF

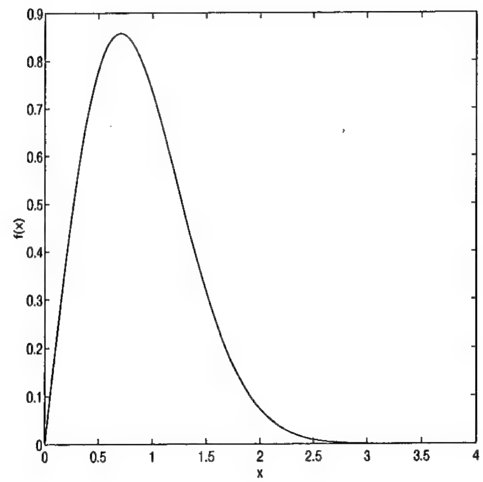

Figure 3.3 Weibull(0,1,2) PDF

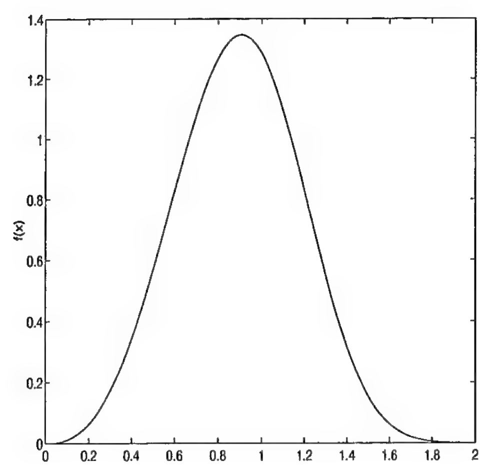

Figure 3.4 Weibull(0,1,3.5) PDF

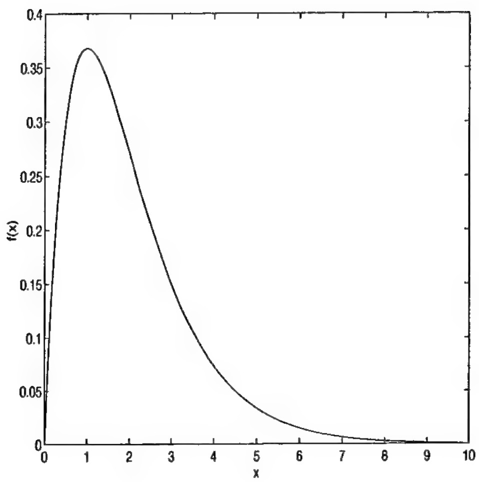

Figure 3.5 Gamma(0,1,2) PDF

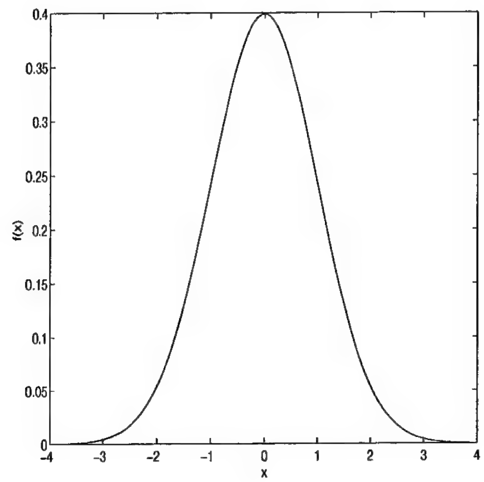

Figure 3.6 Normal(0,1) PDF

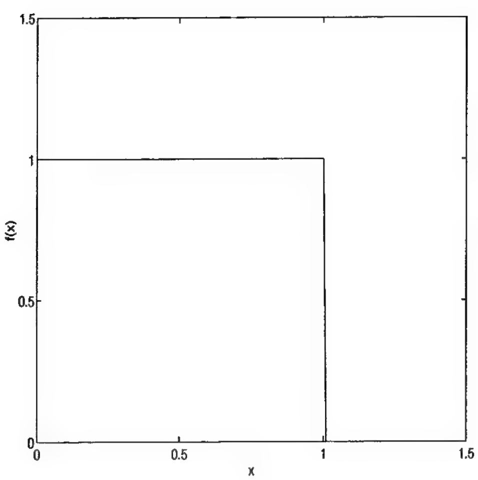

Figure 3.7 Uniform(0,1) PDF

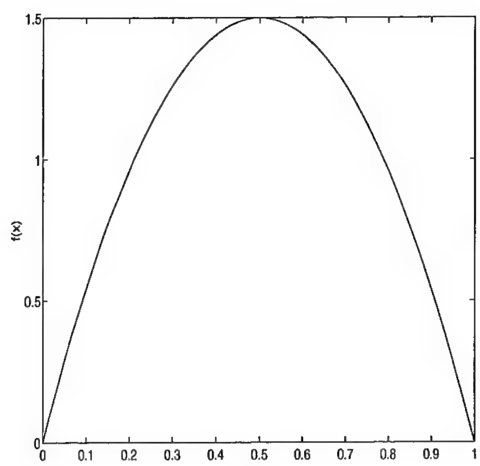

Figure 3.8 Beta(2,2) PDF

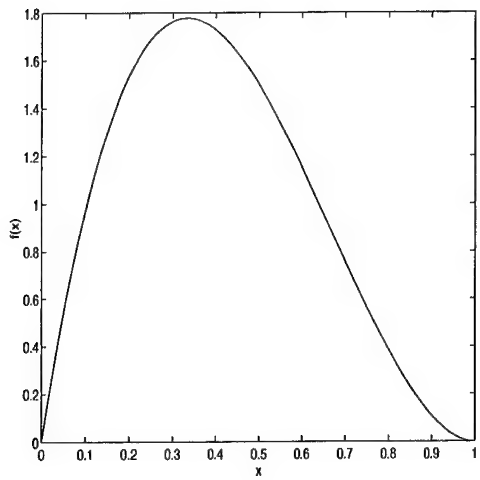

Figure 3.9 Beta(2,3) PDF

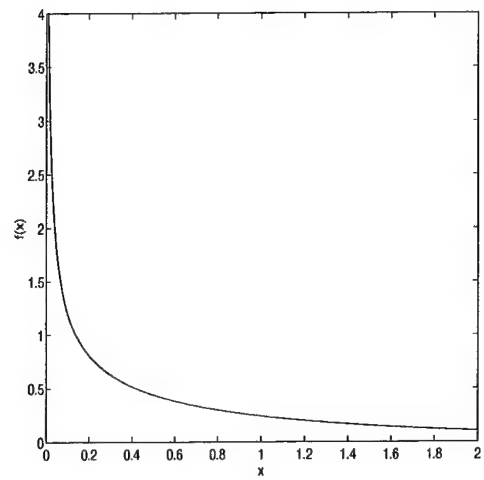

Figure 3.10 Chi-square(1) PDF

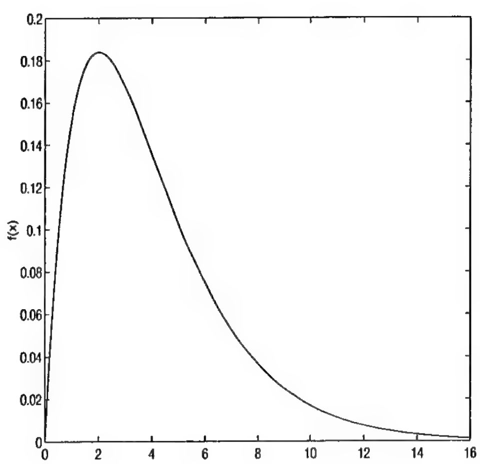

Figure 3.11 Chi-square(4) PDF

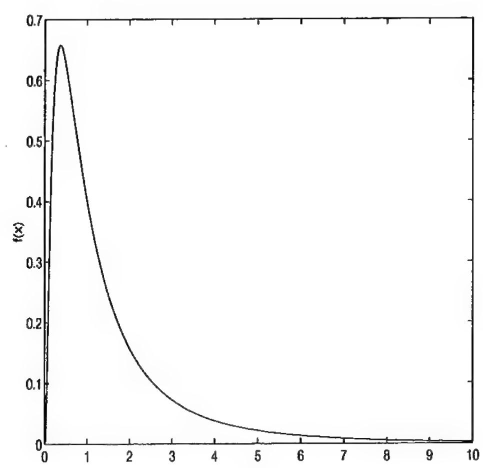

Figure 3.12 Log-normal(0,1) PDF

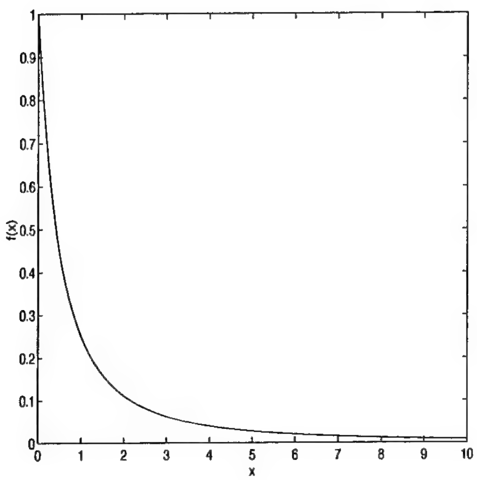

Figure 3.13 Log-logistic(0,1) PDF

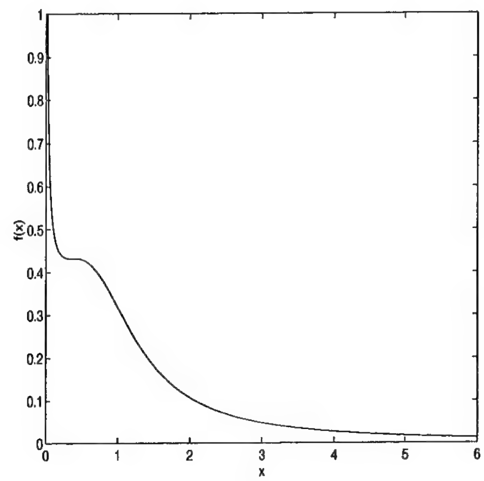

Figure 3.14 Log-Cauchy(0,1) PDF

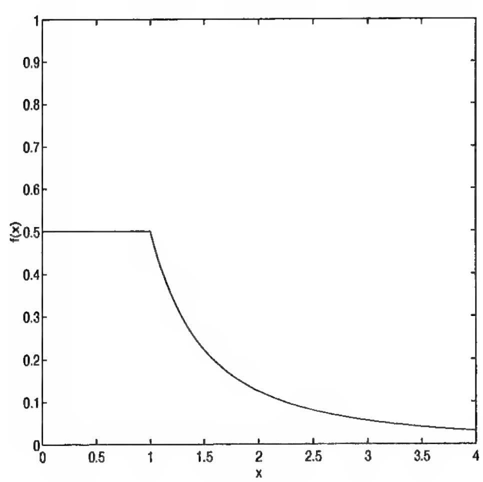

Figure 3.15 Log-double exponential(0,1) PDF

3.5.2 Power Study Process. The power study process is similar to the procedure for determining the critical values. A Monte Carlo simulation with 50,000 replications is used. The details are outlined in the following steps:

1. *Generate random deviates.* For a given sample size n, random deviates are generated from the alternative distribution.

2. *Order the sample.* The n random deviates are sorted in ascending order. This is necessary for the spacings calculations and for generating a censored sample.

3. *Censor the sample.* For a given combination of left and right type II censoring, a "censored sample" is generated by discarding the appropriate portion of the ordered sample. For full samples, this step is omitted.

4. *Compute the test statistics.* The procedure described at the beginning of this chapter is used to determine the values of the modified Cramér-von Mises and Anderson-Darling test statistics.

5. *Compare test statistics to critical values.* The values of A_s^2 and W_s^2 generated from the alternative distribution are compared to the critical values of the null distribution at the 0.1, 0.05, and 0.01 significance levels. The null hypothesis is rejected if the test statistic value of the given sample exceeds the critical value.

6. *Repeat steps 1 through 5.* Steps 1-5 are repeated 50,000 times to obtain accurate power results. The number of times the A_s^2 and W_s^2 statistics lead to rejection of H_0 is counted at each α level.

7. *Calculate the power against H_a.* The power is determined by dividing the number of rejections of H_0 by 50,000 at each α level.

Figure 3.16 summarizes the power study process in flowchart form. This process is repeated for sample sizes 5, 15, and 25. As before, the censoring levels are given in Table 3.1.

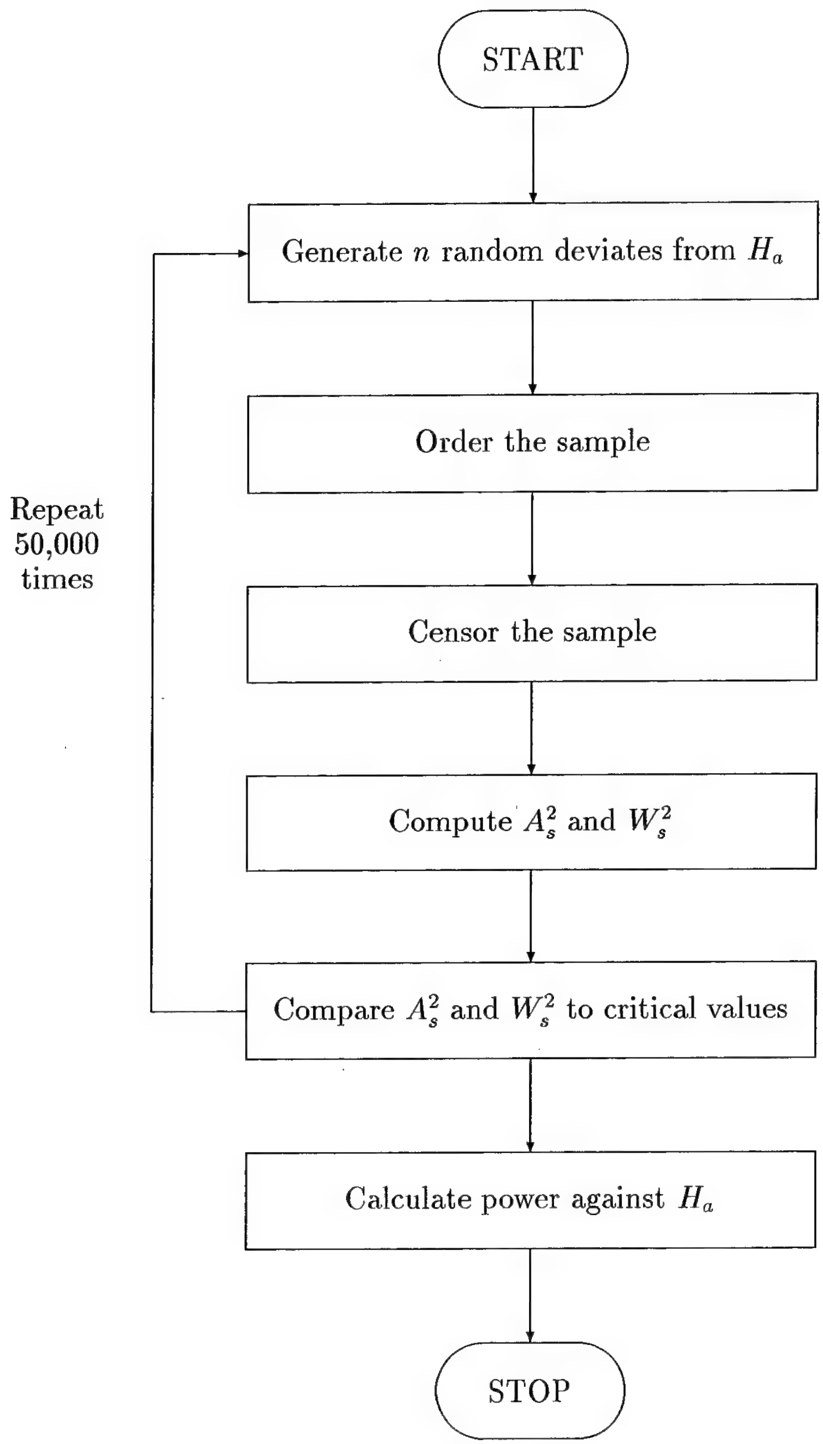

Figure 3.16 Power Study Process

3.6 Random Deviate Generation

Since both the critical value generation and the power study rely on random deviate generation, this section discusses how the deviates are generated. For the Weibull, gamma, normal, uniform, beta, log-normal, and chi-square distributions, random deviates are generated using the functions provided in Matlab 4.0 Statistical Toolbox. These functions are the WEIBRND, GAMRND, NORMRND, UNIFRND, BETARND, LOGNRND, and CHI2RND, respectively. Since Matlab does not have pre-defined functions for the remaining alternatives, they are coded in Matlab from their cumulative distribution functions using the inverse-transformation method, as discussed in Law and Kelton (17:465-503).

3.7 Summary

This chapter described the methodology used to generate the critical values of the new A_s^2 and W_s^2 test statistics and to perform a power study to compare these tests to existing techniques. The next chapter discusses the results that were found.

IV. FINDINGS

4.1 Introduction

This chapter discusses the findings of this research effort. Specifically, the results of each objective outlined in Chapter 1 are presented along with select tables which highlight these results. Complete sets of tables are found in the appendices.

4.2 The Test Statistics

As demonstrated in Chapter 3, the A_s^2 and W_s^2 test statistics are relatively easy to compute since the location and scale parameters of the hypothesized Weibull distribution in the goodness-of-fit test do not need to be known nor estimated. On the other hand, the expected values of a given sample from the null distribution must be available or computed. Originally, the expected values of the ordered spacings used in this thesis were computed with Matlab from Equation 3.3 of this document. However, verification of the computer code, discussed at the end of this chapter, revealed that the computed values diverged from the tabled values found in Harter (13) beyond sample size $n = 30$, regardless of the shape parameter. This is attributed to a problem with computer accuracy. For this reason, a table look-up procedure with Harter's values was implemented in the computer code used throughout this thesis, as mentioned previously in the Methodology section. This greatly reduced the computer simulation time since the table look-up is much more efficient than evaluating Equation 3.3 to get the expected values of the ordered spacings.

In comparing the two statistics we note that W_s^2 has the following advantage over A_s^2. In real data, ties may occur due to rounding or measurement error. This leads to spacings which are zero. If the ties are at the extreme observations, A_s^2 will be infinite. This may lead to rejection of an approximate null distribution (2). The W_s^2 statistic does not have this drawback.

4.3 Critical Values

Critical values of the A_s^2 and W_s^2 test statistics were obtained via Monte Carlo simulations with 100,000 iterations each. Complete tabled values for combinations of shape $\beta = 0.5(0.5)4.0$, sample size $n = 5(5)40$, significance level $\alpha = 0.01$, 0.025, 0.05(0.05)0.25, and censoring levels up to 40% from the left and/or the right are available in Appendices A and B. Use of these tables is shown first, followed by some results and a discussion of the findings.

4.3.1 Use of Tables.

This section describes the use of the tabled critical values presented in this chapter and in the appendices of this thesis. As such, it is a guide to performing the goodness-of-fit tests developed in this research. These tests are summarized in the following procedure:

1. Specify the shape parameter for the hypothesized theoretical Weibull distribution based on previous knowledge or analysis of the system under study.
2. Select and compute either the A_s^2 or W_s^2 test statistic using the algorithm introduced in Chapter 3, Section 3.2.
3. Specify the significance level, α, of the test. Remember that the level of significance is the probability of incorrectly rejecting the null hypothesis.
4. Extract the critical value from the appropriate table based on the test statistic employed, the shape parameter of the hypothesized distribution, the sample size n, and the significance level of the test. If the computed test statistic is from a censored sample, the proper combination of left and right censoring must also be noted.
5. Compare the computed test statistic value to the critical value extracted from the table. If the test statistic is larger than the critical value, reject the null hypothesis that the sample data follows the hypothesized distribution.

For instance, recall the numerical example given in Chapter 3, Section 3.2.1. This is an example of the A_s^2 and W_s^2 test statistic computations from a right-Type II censored sample. The left censoring level is zero and the right censoring level is 0.8, since the largest twenty percent of the original ordered observations were discarded. Now suppose we wish to test the null hypothesis that this censored sample follows a Weibull distribution with shape $\beta = 1$ using the A_s^2 test statistic. Using the procedure outlined above, we must employ the following table of critical values:

Table 4.1 A_s^2 Critical Values for Shape $\beta = 1.0$, Sample size $= 10$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
1.0	10	0.00	1.00	1.248	1.410	1.627	1.953	2.523	3.112	3.939
1.0	10	0.00	0.80	1.243	1.407	1.626	1.946	**2.519**	3.122	3.950
1.0	10	0.00	0.60	1.242	1.405	1.627	1.961	2.550	3.168	4.013
1.0	10	0.20	1.00	1.240	1.402	1.627	1.938	2.523	3.111	3.909
1.0	10	0.20	0.80	1.234	1.405	1.626	1.954	2.547	3.164	4.026
1.0	10	0.40	1.00	1.250	1.419	1.634	1.961	2.533	3.137	3.966

Notice that "sample size" in the table refers to the number of items on test, which is not necessarily the number of observations used in the computation of the test statistic due to censoring. If we wish to test our hypothesis at the $\alpha = 0.05$ significance level, we see from the table that the critical value of this test is 2.519. Recall that the value of our test statistic, A_s^2, was found to be 1.924. Since the test statistic value does not exceed the critical value of the test, we do not not have enough evidence to reject the null hypothesis, and we conclude that this sample does follow a Weibull distribution with shape $\beta = 1$.

4.3.2 Critical Value Results. Looking at any table of critical values generated in this thesis, it is clear that the critical values increase as the significance level of the test decreases. This is expected since the acceptance region of the test must get larger as the probability of a Type I error is reduced. There are, however, some more interesting properties to note about the critical values of these two tests.

These properties are discussed in relation to the shape parameter and in relation to the sample size.

When the shape parameter $\beta < 1$, the critical values of both A_s^2 and W_s^2 monotonically increase as sample size increases. This is true of both full and censored samples. Yet further trends are noted for censored samples. The critical values of a left censored sample are less than those from a right censored sample of the same size and same censoring proportion, and greater than those of a left and right censored sample of the same size and censoring proportion. Censoring levels L = 0.00:R = 0.60, L = 0.40:R = 1.00, and L = 0.20:R = 0.80 all have an effective sample size of 6, but the critical values differ as just explained. When $\beta > 1$, the exact opposite trends of A_s^2 and W_s^2 are observed. The critical values monotonically decrease as sample size increases for both full and censored samples. Furthermore, the critical values of a left censored sample are greater than those from a right censored sample of the same size and censoring proportion, and less than those of a left and right censored sample of the same size and censoring proportion. When $\beta = 1$, however, the trends are not as distinct. In this case, the critical values exhibit some properties of both the other cases mentioned. Tables 4.2 and 4.3 show the behavior of the critical values for full samples at the $\alpha = 0.05$ significance level. Overall trends of critical values versus sample size are shown graphically in Figures 4.1 and 4.3. Interestingly enough, these same observations are noted by Bush (5) is his work with the Anderson-Darling and Cramér-von Mises goodness-of-fit tests.

For a given sample size, the critical values of A_s^2 and W_s^2 decrease, in general, as the shape parameter increases. Between shapes $\beta = 0$ and $\beta = 1$, the critical values decrease dramatically. Between shapes $\beta = 1$ and $\beta = 2$, the decrease is more moderate. Beyond shape $\beta = 2$, the critical values of both tests appear to be approaching a limit. This is evident from the slight oscillations of the critical values. The Monte Carlo variability of the experiment may be the cause of these

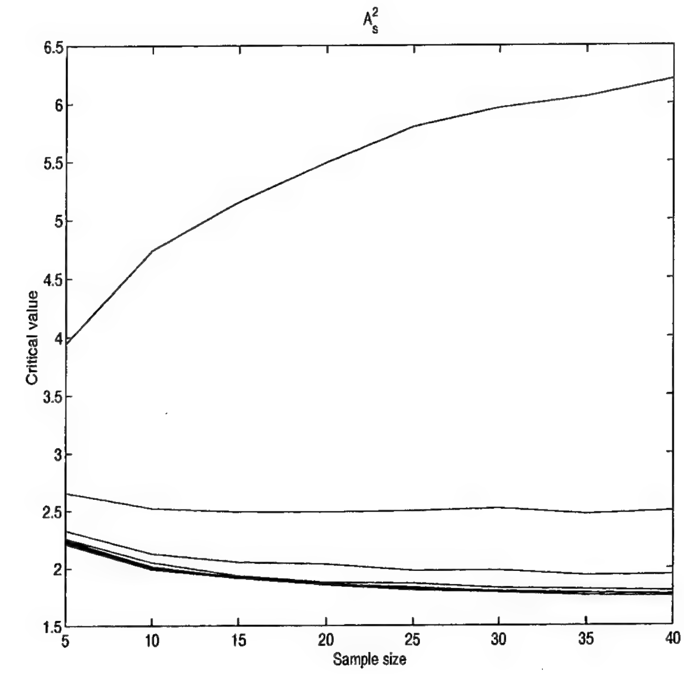

Figure 4.1 A_s^2 Critical Values vs. Sample Size, $\alpha = 0.05$

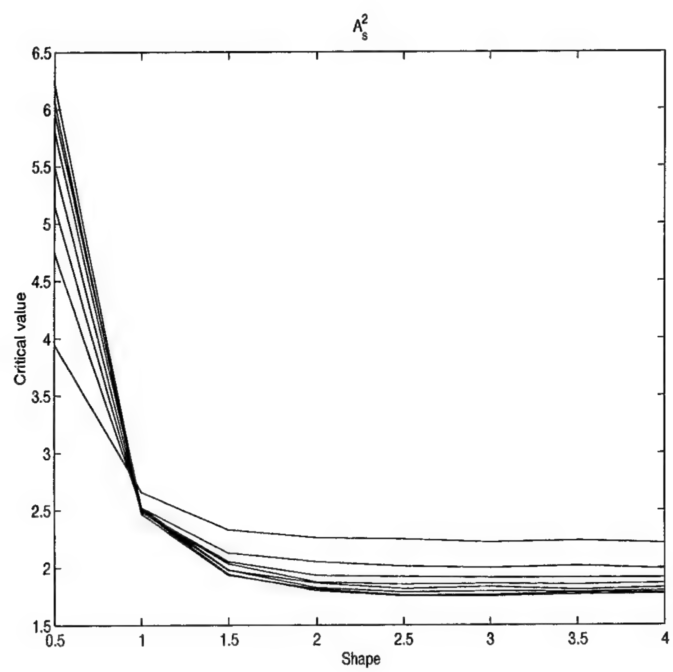

Figure 4.2 A_s^2 Critical Values vs. Shape, $\alpha = 0.05$

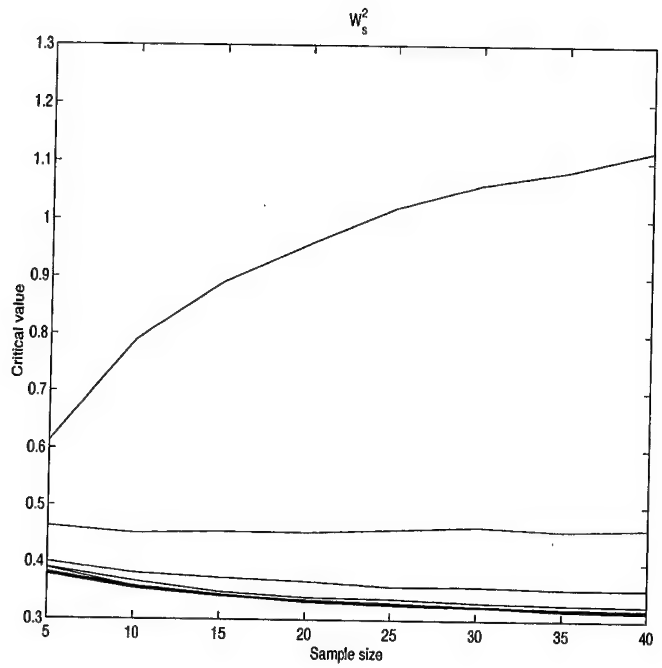

Figure 4.3 W_s^2 Critical Values vs. Sample Size, $\alpha = 0.05$

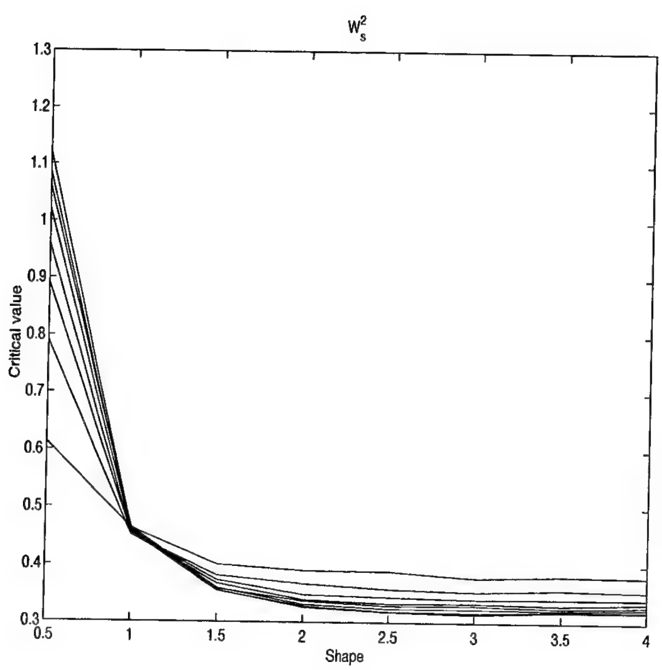

Figure 4.4 W_s^2 Critical Values vs. Shape, $\alpha = 0.05$

Table 4.2 A_s^2 Critical Values for Full Samples, $\alpha = 0.05$

Sample size	Shape parameter β 0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
5	3.944	2.659	2.329	2.260	2.248	2.220	2.235	2.213
10	4.739	2.523	2.127	2.050	2.011	1.997	2.015	1.989
15	5.153	2.492	2.052	1.932	1.921	1.913	1.913	1.912
20	5.493	2.489	2.034	1.874	1.855	1.860	1.853	1.865
25	5.799	2.499	1.977	1.866	1.816	1.833	1.807	1.825
30	5.961	2.520	1.979	1.826	1.786	1.793	1.789	1.800
35	6.058	2.471	1.934	1.814	1.760	1.763	1.781	1.783
40	6.209	2.498	1.942	1.800	1.752	1.751	1.766	1.772

Table 4.3 W_s^2 Critical Values for Full Samples, $\alpha = 0.05$

Sample size	Shape parameter β 0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
5	0.613	0.463	0.400	0.390	0.389	0.378	0.382	0.380
10	0.790	0.451	0.381	0.367	0.358	0.354	0.358	0.355
15	0.891	0.454	0.373	0.348	0.344	0.341	0.342	0.342
20	0.958	0.453	0.367	0.339	0.333	0.334	0.331	0.335
25	1.020	0.458	0.358	0.337	0.329	0.330	0.326	0.329
30	1.060	0.464	0.359	0.331	0.324	0.324	0.323	0.325
35	1.084	0.457	0.355	0.328	0.319	0.317	0.321	0.321
40	1.119	0.461	0.355	0.326	0.318	0.315	0.320	0.320

fluctuations. Again, Tables 4.2 and 4.3 show these behaviors. Overall trends of critical values versus shape parameter are shown graphically in Figures 4.2 and 4.4,

Overall, the A_s^2 and W_s^2 tests have critical values with predictable behaviors. This is a nice feature in a goodness-of-fit test. The inherent stability of the critical values in a given range of shapes and/or sample sizes allows for safe interpolations between these values.

4.4 Power Study

As defined in Chapter 1, the power of a goodness-of-fit test is the probability of rejecting the null hypothesis when a particular alternative hypothesis is true. Recall that the power of the tests developed in this thesis concern the hypotheses

H_0: a random sample of n X-values follows a Weibull distribution with known shape parameter β, versus

H_a: the random sample of n X-values is not Weibull with shape β.

For a good omnibus test, we seek power values as close to 1.0 as possible for any distribution that is not the Weibull distribution specified in the null hypothesis. Three null distributions and fourteen alternative distributions were considered in the power study of W_s^2 and A_s^2. The power of these two tests was computed via Monte Carlo simulations with 50,000 iterations each. Complete tabled values for combinations of shape $\beta = 1.0, 2.0, 3.5$, sample size $n = 5, 15, 25$, significance level $\alpha = 0.10, 0.05, 0.01$, and censoring levels up to 40% from the left and/or the right are available in Appendices C, D, E, F, G, and H. Use of these tables is shown first, followed by some results and a discussion of the findings. At this time it is important to recall from Chapter 2 that EDF tests essentially measure how closely the transformed $Z_{(i)}$ values, computed from the sample (see Chapter 3), follow a uniform distribution. Thus in all cases in this power study, departures from the hypothesized distribution translate to Z-values that diverge from uniformity. In other words, the closer the spacings of the alternative distribution match the spacings of the null distribution, the more uniform the transformed Z-values will be and the lower the power of the test will be, in general. In fact, when the distribution of H_0 and H_a are the same, the power of the test should be equal to the significance level of the test. This will be discussed further in the *Validation* section at the end of this chapter.

4.4.1 Use of Tables. This section describes the use of the power tables presented in this chapter and in the appendices of this thesis. These tables allow the analyst to see how good the W_s^2 or A_s^2 test is against specific alternative distributions. For illustration purposes, consider Table 4.4. In this power test, a random sample

Table 4.4 Example Power of W_s^2

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Lognormal(0,1)	15	0.00	1.00	0.755	0.679	0.519
			0.00	0.80	0.418	0.312	0.151
			0.00	0.60	0.222	0.141	0.046
			0.20	1.00	0.557	0.463	0.295
			0.20	0.80	0.232	0.149	0.050
			0.40	1.00	0.381	0.288	0.154

of size 15 was repeatedly drawn from a log-normal(0,1) distribution and was tested using the W_s^2 goodness-of-fit test to be Weibull with shape $\beta = 2$. For each sample drawn, the W_s^2 test statistic, computed from the log-normal data, was compared to the critical value of the test for each level of significance and degree of censoring shown in Table 4.4. If the test statistic was greater than the critical value, the hypothesized Weibull distribution was rejected as the population distribution of the sample. The total number of rejections divided by the total number of test repetitions is the power of the test as shown in the table. In Table 4.4, we see that the W_s^2 test does a fairly good job of discriminating a log-normal distribution from this particular Weibull distribution for a full sample of size 15. In this case, nearly 70% of the 50,000 samples tested were rejected as being Weibull with shape $\beta = 2$, at the 0.05 significance level. With the introduction of moderate censoring, though, the power is reduced significantly. At the same significance level, censoring 20% of the sample data from the right (R = 0.80) results in a power for the test that is less than half of what was achieved for the full sample.

4.4.2 Power Results for Shape Parameter β = 1.0. The Weibull distribution with shape $\beta = 1.0$ is the most skewed distribution considered in this power study, as seen in Figure 3.2 of Chapter 3, Section 3.5. In fact, the Weibull distribution with this particular shape parameter reduces to an exponential distribution. A graphical analysis of the alternative distributions reveals that this Weibull distribution differs greatly from all of the alternatives with the exception of the gamma($\beta =$ 2), chi-square(4), and log-normal(0,1) distributions. Therefore, we expect the power of both W_s^2 and A_s^2 to be poor against these alternatives. Tables 4.5, 4.6, and 4.7 confirm this expectation. These tables show the achieved power of the W_s^2 and A_s^2 tests for full samples (no censoring) of size 5, 15, and 25, respectively. Complete tables with various degrees of censoring are given in Appendices C and F. Clearly, as the sample size increases, the power of the each test increases, regardless of the alternative distribution. This is an intuitive result because more information about the underlying distribution is available from a sample of size 25 than from a sample of size 15 or 5. For each sample size, W_s^2 and A_s^2 achieve high power against symmetric, or nearly symmetric alternatives, such as the normal(0,1) distribution and the Weibull($\beta = 3.5$) distribution, and against highly skewed-alternatives like the log-Cauchy(0,1) distribution. This is not surprising. For example, the normal distribution is symmetric and is unbounded in both directions. Hence, we would expect the magnitude of the spacings of this distribution to be small near the center, or mean of the distribution, and large in the tails. Since the null distribution is skewed right and has only one tail, the spacings, in comparison, are grossly different. The log-Cauchy on the other hand, although skewed like the null distribution, has a thicker tail. Therefore, we would expect the magnitude of the spacings in that part of the distribution to be much greater than those of the null distribution. Similar logic can be extended to other alternatives as well. Again, Tables 4.5, 4.6, and 4.7 show the power of the W_s^2 and A_s^2 test against all the alternatives considered in this research effort.

Table 4.5 Power of A_s^2 for H_0: Weibull($\beta = 1$), Sample size = 5

Alternative Distribution, H_a	$\alpha = 0.10$ W_s^2	A_s^2	$\alpha = 0.05$ W_s^2	A_s^2	$\alpha = 0.01$ W_s^2	A_s^2
Weibull($\beta = 1$)	.102	.100	.051	.051	.009	.010
Weibull($\beta = 2$)	.180	.169	.099	.095	.022	.021
Weibull($\beta = 3.5$)	.267	.256	.163	.154	.041	.038
Gamma($\beta = 2$)	.121	.116	.063	.060	.012	.012
Normal(0,1)	.270	.256	.167	.159	.043	.042
Uniform(0,1)	.220	.238	.137	.146	.039	.041
Beta(2,2)	.242	.241	.148	.147	.036	.036
Beta(2,3)	.198	.192	.112	.109	.025	.025
Chi-square(1)	.151	.176	.090	.106	.028	.034
Chi-square(4)	.122	.119	.064	.061	.013	.013
Log-normal(0,1)	.129	.121	.070	.065	.016	.015
Log-logistic(0,1)	.295	.301	.220	.223	.116	.117
Log-double exp(0,1)	.245	.232	.173	.164	.082	.078
Log-Cauchy(0,1)	.484	.499	.429	.442	.347	.360

For censored samples, the power results parallel the findings found from full samples. Consider Table 4.8 for instance. Here we see the power of A_s^2 for testing the fit of Weibull($\beta = 2$) data to our null distribution. At this particular sample size, the power of the test is good when no censoring schemes are employed. At the $\alpha = 0.05$ significance level, we are able to reject approximately 75% of the samples as being Weibull with shape $\beta = 1$. With the introduction of moderate censoring, say 20% from the right, the power of the test diminishes as expected since we have effectively reduced our sample size to 20. This decrease in power, though, is not great because H_0 and H_a do not differ tremendously in the right tail portions of their distributions, as shown in Figure 4.5. However, notice what happens with the introduction of left censoring. The power of the test reduces by nearly a factor of three. With left censoring, we have lost information in the area where the two distributions differ the most. Thus, it is more difficult for the test to discriminate between them. Again, similar logic can be extended to other alternatives as well.

Table 4.6 Power of A_s^2 for H_0: Weibull($\beta = 1$), Sample size = 15

Alternative Distribution, H_a	$\alpha = 0.10$ W_s^2	A_s^2	$\alpha = 0.05$ W_s^2	A_s^2	$\alpha = 0.01$ W_s^2	A_s^2
Weibull($\beta = 1$)	.102	.101	.051	.051	.010	.010
Weibull($\beta = 2$)	.566	.561	.430	.427	.199	.194
Weibull($\beta = 3.5$)	.836	.840	.746	.757	.519	.527
Gamma($\beta = 2$)	.239	.228	.149	.143	.043	.041
Normal(0,1)	.841	.842	.760	.765	.548	.551
Uniform(0,1)	.630	.714	.493	.591	.242	.325
Beta(2,2)	.773	.801	.654	.695	.381	.422
Beta(2,3)	.634	.649	.498	.517	.244	.255
Chi-square(1)	.427	.492	.320	.389	.155	.210
Chi-square(4)	.243	.231	.152	.144	.047	.043
Log-normal(0,1)	.228	.210	.154	.139	.066	.056
Log-logistic(0,1)	.764	.763	.704	.705	.579	.578
Log-double exp(0,1)	.576	.552	.505	.481	.389	.364
Log-Cauchy(0,1)	.891	.898	.870	.877	.831	.835

Table 4.7 Power of A_s^2 for H_0: Weibull($\beta = 1$), Sample size = 25

Alternative Distribution, H_a	$\alpha = 0.10$ W_s^2	A_s^2	$\alpha = 0.05$ W_s^2	A_s^2	$\alpha = 0.01$ W_s^2	A_s^2
Weibull($\beta = 1$)	.101	.100	.052	.052	.010	.010
Weibull($\beta = 2$)	.843	.844	.751	.755	.515	.517
Weibull($\beta = 3.5$)	.982	.984	.962	.968	.882	.895
Gamma($\beta = 2$)	.412	.400	.289	.281	.110	.104
Normal(0,1)	.983	.984	.967	.970	.897	.904
Uniform(0,1)	.884	.943	.796	.888	.544	.693
Beta(2,2)	.968	.980	.933	.955	.788	.842
Beta(2,3)	.903	.919	.831	.856	.605	.643
Chi-square(1)	.640	.711	.532	.616	.318	.413
Chi-square(4)	.410	.398	.287	.279	.112	.107
Log-normal(0,1)	.288	.269	.204	.187	.097	.085
Log-logistic(0,1)	.920	.920	.891	.891	.812	.812
Log-double exp(0,1)	.732	.716	.673	.656	.569	.544
Log-Cauchy(0,1)	.976	.979	.970	.974	.956	.959

Table 4.8 Example Power of W_s^2 for Censored Samples

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Weibull($\beta = 2$)	25	0.00	1.00	0.844	0.755	0.517
			0.00	0.80	0.645	0.522	0.273
			0.00	0.60	0.436	0.310	0.128
			0.20	1.00	0.400	0.280	0.102
			0.20	0.80	0.213	0.130	0.040
			0.40	1.00	0.203	0.122	0.033

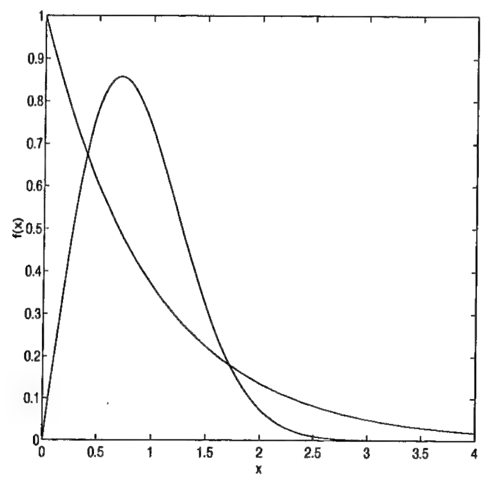

Figure 4.5 Weibull(0,1,1) vs. Weibull(0,1,2)

Overall, both W_s^2 and A_s^2 have similar power properties for testing the alternatives considered in this study against the Weibull distribution with shape $\beta = 1$. Furthermore, W_s^2 and A_s^2 have nearly equal power against a given alternative, sample size, degree of censoring, and significance level. Yet, there is considerable improvement with the A_s^2 test for the uniform(0,1) and chi-square(1) distributions at sample sizes 15 and 25.

4.4.3 Power Results for Shape Parameter $\beta = 3.5$. The Weibull distribution with shape $\beta = 3.5$ is nearly symmetric, as seen in Figure 3.4 of Chapter 3, Section 3.5. In this case, graphical analysis of the alternative distributions reveals that this Weibull distribution differs greatly from the highly skewed alternatives and

only slighty from symmetric and nearly symmetric alternatives. This is essentially opposite to what we saw in the previous section. Therefore, following the same logic, we would expect both W_s^2 and A_s^2 to achieve very high power against the log-Cauchy(0,1) and log-logistic(0,1) distributions and poor power against the normal(0,1) and beta(2,2) distributions, for example. Tables 4.9, 4.10, and 4.11 confirm these expectations. These tables show the achieved power of the W_s^2 and A_s^2 tests for full samples (no censoring) of size 5, 15, and 25, respectively. Complete tables with various degrees of censoring are given in Appendices E and H. As before, the power of each test increases with respect to sample size, regardless of the alternative distribution. We see, though, even at sample size $n = 5$, each test rejects approximately 60% of the 50,000 log-Cauchy samples tested as being Weibull with shape $\beta = 3.5$. On the other hand, at sample size $n = 25$, W_s^2 and A_s^2 achieve no more than 5% power against the beta(2,2) distribution. Again, these are intuitive results considering Figures 4.6 and 4.7.

For censored samples, the power results essentially parallel the findings found from full samples. In general, the introduction of censoring diminshes the power of the tests, as explained in the previous section. However, there are a few cases that result in marginally higher power for censored samples, as shown in Tables 4.12 and 4.13. This phenomenon occurs with the symmetric alternatives. Consider Figure 4.7 one more time. Censoring one side results in a skewness that can be identified with the tests. Furthermore, notice that the power of the A_s^2 test in this example is marginally better than the W_2^2 test. This is not surprising. Recall from Chapter 2 that the Anderson-Darling test statistic is designed to emphasize differences in the tails of the distributions under study. Here, we see A_s^2 bring out the difference between the long-tailed Weibull expected values and the short-tailed beta distribution.

Overall, both W_s^2 and A_s^2 have similar power properties for testing the alternatives considered in this study against the Weibull distribution with shape $\beta = 3.5$.

Table 4.9 Power of A_s^2 for H_0: Weibull($\beta = 3.5$), Sample size = 5

Alternative Distribution, H_a	$\alpha = 0.10$ W_s^2	A_s^2	$\alpha = 0.05$ W_s^2	A_s^2	$\alpha = 0.01$ W_s^2	A_s^2
Weibull($\beta = 1$)	.260	.272	.170	.173	.060	.058
Weibull($\beta = 2$)	.117	.116	.061	.059	.013	.012
Weibull($\beta = 3.5$)	.099	.098	.049	.049	.009	.010
Gamma($\beta = 2$)	.183	.179	.109	.101	.028	.027
Normal(0,1)	.105	.104	.054	.052	.011	.011
Uniform(0,1)	.103	.123	.054	.062	.011	.013
Beta(2,2)	.094	.101	.047	.049	.010	.011
Beta(2,3)	.099	.103	.050	.052	.011	.011
Chi-square(1)	.407	.465	.302	.352	.149	.179
Chi-square(4)	.182	.179	.109	.103	.029	.027
Log-normal(0,1)	.350	.356	.254	.253	.112	.109
Log-logistic(0,1)	.555	.593	.463	.498	.307	.328
Log-double exp(0,1)	.463	.470	.379	.380	.239	.236
Log-Cauchy(0,1)	.650	.673	.586	.610	.481	.504

Table 4.10 Power of A_s^2 for H_0: Weibull($\beta = 3.5$), Sample size = 15

Alternative Distribution, H_a	$\alpha = 0.10$ W_s^2	A_s^2	$\alpha = 0.05$ W_s^2	A_s^2	$\alpha = 0.01$ W_s^2	A_s^2
Weibull($\beta = 1$)	.795	.832	.709	.755	.504	.558
Weibull($\beta = 2$)	.223	.223	.139	.139	.044	.043
Weibull($\beta = 3.5$)	.100	.100	.050	.050	.009	.010
Gamma($\beta = 2$)	.551	.562	.441	.451	.238	.242
Normal(0,1)	.120	.116	.063	.059	.014	.013
Uniform(0,1)	.111	.183	.052	.099	.009	.022
Beta(2,2)	.075	.092	.034	.044	.005	.008
Beta(2,3)	.115	.126	.059	.065	.011	.013
Chi-square(1)	.955	.979	.925	.963	.819	.899
Chi-square(4)	.551	.563	.439	.450	.238	.243
Log-normal(0,1)	.896	.910	.847	.866	.714	.741
Log-logistic(0,1)	.988	.992	.979	.987	.948	.965
Log-double exp(0,1)	.947	.951	.926	.929	.866	.872
Log-cauchy(0,1)	.988	.992	.983	.988	.968	.975

Table 4.11 Power of A_s^2 for H_0: Weibull($\beta = 3.5$), Sample size = 25

Alternative Distribution, H_a	$\alpha = 0.10$ W_s^2	$\alpha = 0.10$ A_s^2	$\alpha = 0.05$ W_s^2	$\alpha = 0.05$ A_s^2	$\alpha = 0.01$ W_s^2	$\alpha = 0.01$ A_s^2
Weibull($\beta = 1$)	.962	.976	.932	.957	.838	.887
Weibull($\beta = 2$)	.347	.355	.243	.250	.100	.099
Weibull($\beta = 3.5$)	.102	.103	.052	.053	.011	.011
Gamma($\beta = 2$)	.795	.814	.707	.733	.512	.534
Normal(0,1)	.130	.128	.069	.067	.017	.015
Uniform(0,1)	.175	.327	.087	.201	.017	.056
Beta(2,2)	.080	.103	.036	.050	.005	.009
Beta(2,3)	.161	.177	.088	.102	.022	.025
Chi-square(1)	.999	1.000	.997	.999	.985	.997
Chi-square(4)	.793	.812	.708	.732	.510	.532
Log-normal(0,1)	.990	.993	.981	.987	.949	.962
Log-logistic(0,1)	1.000	1.000	1.000	1.000	.999	1.000
Log-double exp(0,1)	.996	.997	.993	.995	.985	.987
Log-Cauchy(0,1)	.999	1.000	.999	1.000	.999	.999

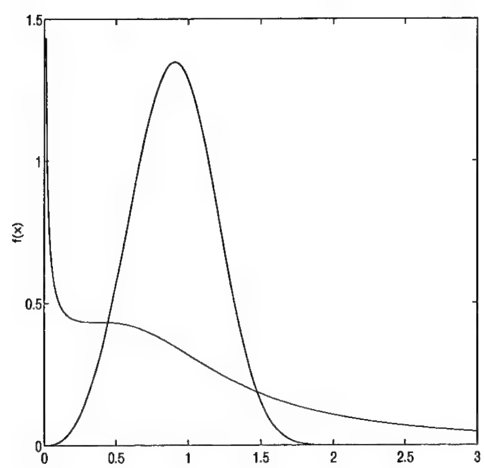

Figure 4.6 Weibull(0,1,3.5) vs. Log-Cauchy(0,1)

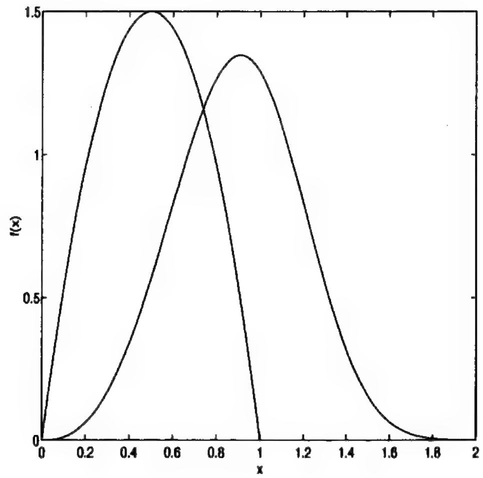

Figure 4.7 Beta(2,2) vs. Weibull(0,1,3.5)

Table 4.12 Example Power of W_s^2 for Censored Samples

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Beta(2,2)	25	0.00	1.00	0.080	0.036	0.005
			0.00	0.80	0.114	0.060	0.014
			0.00	0.60	0.124	0.068	0.016
			0.20	1.00	0.105	0.053	0.012
			0.20	0.80	0.093	0.047	0.009
			0.40	1.00	0.121	0.065	0.016

Furthermore, W_s^2 and A_s^2 have nearly equal power against a given alternative, sample size, degree of censoring, and significance level. Yet, there is considerable improvement with the A_s^2 test for the chi-square(1) distribution at sample sizes $n = 5$ and 15, the Weibull($\beta = 1$) distribution at $n = 15$, and the uniform(0,1) distribution at $n = 15$ and 25.

4.4.4 Power Results for Shape Parameter $\beta = 2.0$. The Weibull distribution with shape $\beta = 2$ is slightly skewed-right, as seen in Figure 3.3 of Chapter 3, Section 3.5. In this case, graphical analysis of the alternative distributions reveals that this Weibull distribution differs moderately to greatly from all the alternatives, except for the beta(2,3) distribution, as shown in Figure 4.8. For illustration pur-

Table 4.13 Example Power of A_s^2 for Censored Samples

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Beta(2,2)	25	0.00	1.00	0.103	0.050	0.009
			0.00	0.80	0.126	0.065	0.015
			0.00	0.60	0.132	0.070	0.017
			0.20	1.00	0.118	0.060	0.012
			0.20	0.80	0.098	0.049	0.010
			0.40	1.00	0.129	0.069	0.015

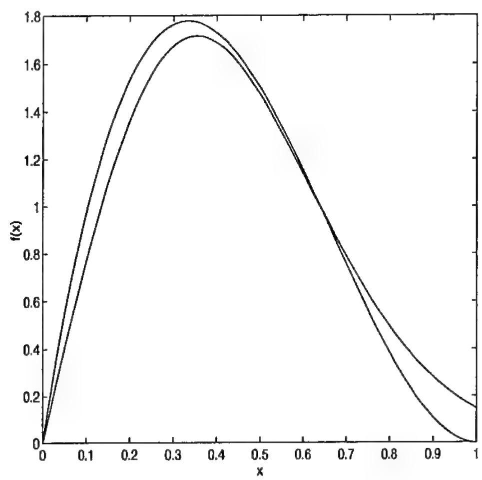

Figure 4.8 Weibull(0,4,2) vs. Beta(2,3)

poses, the scale parameter of the null distribution in the figure has been adjusted to show how close the probabilty densities, and thus the spacings, match.

Following the logic from before, we expect both W_s^2 and A_s^2 to have poor power against the beta(2,3) alternative. Tables 4.14, 4.15, and 4.16 confirm this expectation. These tables show the achieved power of the W_2^s and A_s^2 tests for full samples (no censoring) of size 5, 15, and 25, respectively. Complete tables with various degrees of censoring are given in Appendices D and G. Even with a full sample of size 25, the tests yield only about 6% power against the beta(2,3) alternative. Again, similar logic can be extended to the other distributions as well. In general, since the Weibull distribution with shape $\beta = 2$ is skewed less than the Weibull(β

Table 4.14 Power of A_s^2 for H_0: Weibull($\beta = 2$), Sample size = 5

Alternative Distribution, H_a	$\alpha = 0.10$ W_s^2	A_s^2	$\alpha = 0.05$ W_s^2	A_s^2	$\alpha = 0.01$ W_s^2	A_s^2
Weibull($\beta = 1$)	.191	.197	.113	.115	.033	.032
Weibull($\beta = 2$)	.099	.098	.048	.049	.010	.010
Weibull($\beta = 3.5$)	.119	.115	.060	.059	.013	.012
Gamma($\beta = 2$)	.130	.126	.068	.064	.015	.014
Normal(0,1)	.125	.119	.065	.062	.015	.013
Uniform(0,1)	.114	.132	.061	.069	.014	.015
Beta(2,2)	.107	.110	.054	.055	.011	.011
Beta(2,3)	.096	.097	.045	.047	.009	.009
Chi-square(1)	.328	.376	.232	.273	.106	.124
Chi-square(4)	.132	.129	.070	.066	.015	.015
Log-normal(0,1)	.273	.272	.184	.179	.073	.066
Log-logistic(0,1)	.486	.514	.396	.420	.248	.259
Log-double exp(0,1)	.392	.393	.312	.308	.187	.180
Log-Cauchy(0,1)	.604	.623	.539	.562	.441	.459

Table 4.15 Power of A_s^2 for H_0: Weibull($\beta = 2$), Sample size = 15

Alternative Distribution, H_a	$\alpha = 0.10$ W_s^2	A_s^2	$\alpha = 0.05$ W_s^2	A_s^2	$\alpha = 0.01$ W_s^2	A_s^2
Weibull($\beta = 1$)	.573	.600	.464	.494	.270	.292
Weibull($\beta = 2$)	.099	.100	.050	.050	.009	.010
Weibull($\beta = 3.5$)	.258	.251	.166	.161	.058	.054
Gamma($\beta = 2$)	.272	.269	.184	.180	.073	.067
Normal(0,1)	.289	.278	.194	.184	.076	.068
Uniform(0,1)	.192	.268	.111	.169	.031	.053
Beta(2,2)	.193	.208	.113	.123	.031	.033
Beta(2,3)	.102	.109	.051	.055	.011	.011
Chi-square(1)	.886	.926	.828	.887	.673	.769
Chi-square(4)	.270	.266	.184	.180	.072	.067
Log-normal(0,1)	.755	.762	.679	.689	.519	.522
Log-logistic(0,1)	.965	.973	.949	.959	.897	.915
Log-double exp(0,1)	.879	.878	.844	.842	.762	.758
Log-Cauchy(0,1)	.973	.977	.964	.970	.942	.950

Table 4.16 Power of A_s^2 for H_0: Weibull($\beta = 2$), Sample size = 25

Alternative Distribution, H_a	$\alpha = 0.10$ W_s^2	A_s^2	$\alpha = 0.05$ W_s^2	A_s^2	$\alpha = 0.01$ W_s^2	A_s^2
Weibull($\beta = 1$)	.808	.838	.727	.765	.531	.577
Weibull($\beta = 2$)	.099	.100	.049	.050	.010	.010
Weibull($\beta = 3.5$)	.394	.391	.281	.278	.121	.116
Gamma($\beta = 2$)	.395	.394	.291	.289	.139	.133
Normal(0,1)	.441	.435	.334	.328	.165	.154
Uniform(0,1)	.295	.439	.182	.301	.058	.113
Beta(2,2)	.296	.326	.189	.213	.061	.070
Beta(2,3)	.116	.126	.059	.066	.013	.015
Chi-square(1)	.989	.995	.977	.991	.932	.970
Chi-square(4)	.397	.396	.293	.292	.141	.137
Log-normal(0,1)	.931	.937	.897	.904	.798	.808
Log-logistic(0,1)	.999	.999	.997	.998	.992	.994
Log-double exp(0,1)	.976	.976	.966	.966	.939	.938
Log-Cauchy(0,1)	.998	.999	.997	.998	.995	.996

= 1) distribution and skewed more than the Weibull($\beta = 3.5$) distribution, we see that the results in this section follow from the results of the previous two sections. For example, consider a skewed alternative, such as the chi-square(1) distribution in Table 4.15. We find the powers of both W_s^2 and A_s^2 to be greater than those attained from the same test with null distribution Weibull($\beta = 1$). This is because the Weibull($\beta = 2$) distribution is skewed less than the Weibull($\beta = 1$) distribution, and the tests are able to reject more of the chi-square(1) samples as being Weibull(β=2). On the other hand, we find the powers of both W_s^2 and A_s^2 in Table 4.15 to be less than those attained from the same test with null distribution Weibull($\beta = 3.5$). This is because the Weibull($\beta = 2$) distribution is skewed more than the Weibull($\beta = 3.5$) distribution, and the tests of the Weibull($\beta = 2$) are not able to reject as many of the chi-square(1) samples.

With the introduction of censored samples, the results are much like before. Therefore, a detailed discussion of the findings is omitted. Again, the results for censored samples are given in the appendices.

Overall, both W_s^2 and A_s^2 have similar power properties for testing the alternatives considered in this study against the Weibull distribution with shape $\beta = 2$. Furthermore, W_s^2 and A_s^2 have nearly equal power against a given alternative, sample size, degree of censoring, and significance level. Yet, like before, A_s^2 is more powerful than W_s^2 against the chi-square(1) distribution at sample sizes $n = 5$ and 15, the Weibull($\beta = 1$) distribution at $n = 15$ and 25, and the uniform(0,1) distribution at $n = 15$ and 25.

4.5 Comparative Power Study

In this section, the powers of W_s^2 and A_s^2 are compared to the powers of other goodness-of-fit tests for the Weibull distribution with known shape parameter. Specifically, comparisons are made to tests developed in prior AFIT masters theses by Bush (5), Coppa (7), and Cortes (8). Recall from Chapter 2 that Bush developed modified EDF tests, namely the Anderson-Darling (A^2) and Cramér-von Mises (W^2) tests in which the location and scale parameters of the hypothesized Weibull distribution were estimated via the method of maximum-likelihood. Similarly, Cortes developed the modified Kolmogorov-Smirnov (KS) test. Whereas Coppa was able to eliminate the need for parameter estimation by employing a test, Z^*, based on the sample spacings of the hypothesized distribution. Powers of W_s^2, A_s^2, Z^*, KS, W^2, and A^2 are compared for null hypotheses H_0:

1. Weibull($\beta = 1$)
2. Weibull($\beta = 3.5$)

versus the following alternatives H_a:

1. Weibull($\beta = 1$)
2. Weibull($\beta = 2$)
3. Weibull($\beta = 3.5$)

4. Gamma($\beta = 2$)

5. Normal(0,1)

6. Uniform(0,1)

7. Beta(2,2)

8. Beta(2,3)

Comparisons to the modified EDF tests (KS,W^2,A^2) are made for full samples (no censoring) of size 5, 15, and 25. While comparisons to Z^* are made only for samples sizes $n = 5$ and 15, since data is not available for $n = 25$.

The null hypothesis with Weibull shape $\beta = 1$ is considered first. Tables 4.17 and 4.18 show the powers of the competing tests at the $\alpha = 0.05$ and 0.01 significance levels, respectively. At the $\alpha = 0.05$ significance level, the W_s^2 and A_s^2 tests achieve considerably higher power, in general, than their prominent EDF competitors, KS, W^2, and A^2. However, Z^* consistantly out-performs all the tests. In some cases, Z^* is able to reject an additional 15% of the samples tested as being Weibull with shape $\beta = 1$. Similar findings are observed at the $\alpha = 0.01$ significance level, as well. However, there is one case in which A_s^2 achieves marginally higher power over Z^*. This is seen when testing uniform(0,1) samples of size 25.

With shape $\beta = 3.5$ in the null hypothesis, the results are quite different. Tables 4.19 and 4.20 show the powers of the competing tests at the $\alpha = 0.05$ and 0.01 significance levels, respectively. For $\alpha = 0.05$, the results are somewhat mixed. Consider power comparisons of W_s^2 and A_s^2 to the other modified EDF tests, first. For $n = 5$, W_s^2 and A_s^2 have nearly equal power compared to KS, W^2, and A^2 for most of the alternatives. Yet for the Weibull($\beta = 1$) and gamma($\beta = 2$) alternatives, the power of W_s^2 and A_s^2 is marginally lower. At sample size $n = 15$, though, the power of W_s^2 and A_s^2 increases for some alternatives and decreases for others. In particular, the powers of W_s^2 and A_s^2 are considerably higher than the competition for the Weibull($\beta = 1$) and Weibull($\beta = 2$) alternatives and W_s^2 is considerably lower

Table 4.17 Power Comparisons for H_0: Weibull($\beta = 1$), $\alpha = 0.05$

Sample size	Alternative Distribution	W_s^2	A_s^2	Z^*	KS	W^2	A^2
5	Weibull($\beta = 1$)	.051	.051	.047	.045	.045	.049
	Weibull($\beta = 2$)	.099	.095	.137	.051	.055	.049
	Weibull($\beta = 3.5$)	.163	.154	.226	.079	.097	.052
	Gamma($\beta = 2$)	.063	.060	.088	.040	.043	.114
	Normal(0,1)	.167	.159	.218	.079	.099	.202
	Uniform(0,1)	.137	.146	.188	.072	.084	.050
	Beta(2,2)	.148	.147	.200	.069	.082	.031
	Beta(2,3)	.112	.109	.161	.057	.066	.014
15	Weibull($\beta = 1$)	.051	.051	.048	.049	.053	.043
	Weibull($\beta = 2$)	.430	.427	.581	.277	.346	.321
	Weibull($\beta = 3.5$)	.746	.757	.852	.568	.667	.624
	Gamma($\beta = 2$)	.149	.143	.240	.099	.116	.122
	Normal(0,1)	.760	.765	.860	.606	.699	.685
	Uniform(0,1)	.493	.591	.650	.328	.446	.384
	Beta(2,2)	.654	.695	.791	.454	.578	.508
	Beta(2,3)	.498	.517	.672	.340	.428	.359
25	Weibull($\beta = 1$)	.052	.052	.052	.057	.056	.047
	Weibull($\beta = 2$)	.751	.755	.877	.575	.700	.655
	Weibull($\beta = 3.5$)	.962	.968	.987	.885	.947	.930
	Gamma($\beta = 2$)	.289	.281	.435	.196	.245	.231
	Normal(0,1)	.967	.970	.984	.904	.950	.938
	Uniform(0,1)	.796	.888	.892	.575	.746	.703
	Beta(2,2)	.933	.955	.974	.771	.899	.867
	Beta(2,3)	.831	.856	.926	.628	.773	.715

than the competition for the uniform(0,1) alternative. For samples of size 25, the power results continue to be mixed. The trends observed for $n = 15$ continue and we see a dramatic increase in the power of both W_s^2 and A_s^2 for the gamma($\beta = 2$) alternative. At this sample size, A_s^2 is able to reject approximately 95% of the gamma($\beta = 2$) samples from being Weibull($\beta = 3.5$), while its closest competitor, A^2, rejects about 47%. Similar findings are observed at the 0.01 significance level, as well.

Table 4.18 Power Comparisons for H_0: Weibull($\beta = 1$), $\alpha = 0.01$

Sample size	Alternative Distribution	W_s^2	A_s^2	Z^*	KS	W^2	A^2
5	Weibull($\beta = 1$)	.009	.010	.011	.009	.008	.007
	Weibull($\beta = 2$)	.022	.021	.033	.003	.002	.004
	Weibull($\beta = 3.5$)	.041	.038	.056	.007	.006	.007
	Gamma($\beta = 2$)	.012	.012	.021	.004	.003	.017
	Normal(0,1)	.043	.042	.062	.003	.004	.061
	Uniform(0,1)	.039	.041	.054	.007	.005	.006
	Beta(2,2)	.036	.036	.051	.005	.005	.004
	Beta(2,3)	.025	.025	.038	.003	.002	.001
15	Weibull($\beta = 1$)	.010	.010	.009	.012	.012	.009
	Weibull($\beta = 2$)	.199	.194	.307	.105	.139	.107
	Weibull($\beta = 3.5$)	.519	.527	.640	.315	.422	.347
	Gamma($\beta = 2$)	.043	.041	.066	.029	.027	.021
	Normal(0,1)	.548	.551	.658	.362	.465	.434
	Uniform(0,1)	.242	.325	.359	.125	.191	.138
	Beta(2,2)	.381	.422	.528	.199	.291	.216
	Beta(2,3)	.244	.255	.375	.134	.177	.122
25	Weibull($\beta = 1$)	.010	.010	.010	.012	.011	.009
	Weibull($\beta = 2$)	.515	.517	.659	.324	.424	.351
	Weibull($\beta = 3.5$)	.882	.895	.933	.717	.828	.781
	Gamma($\beta = 2$)	.110	.104	.190	.069	.083	.062
	Normal(0,1)	.897	.904	.941	.756	.857	.819
	Uniform(0,1)	.544	.693	.684	.303	.449	.361
	Beta(2,2)	.788	.842	.889	.524	.695	.618
	Beta(2,3)	.605	.643	.748	.351	.506	.406

In addition to these findings, Z^* is found to have the weakest power at all sample sizes investigated at shape $\beta = 3.5$. In fact, while the EDF tests are able to reject anywhere from 10% - 20% of the gamma($\beta = 2$) samples ($n = 5$) as being Weibull with shape $\beta = 3.5$, the power of Z^* is approximately 1%. Surprisingly, as the sample size gets larger, the power of Z^* diminishes. This is a counter-intuitive result, since a sample of size 15 contains more information about the underlying distribution than a sample of size 5, as discussed earlier in this chapter. This peculiar behavior of Z^* prompted further investigation of Coppa's research, as discussed in the

Table 4.19 Power Comparisons for H_0: Weibull($\beta = 3.5$), $\alpha = 0.05$

Sample size	Alternative Distribution	W_s^2	A_s^2	Z_e^*	KS	W^2	A^2
5	Weibull($\beta = 1$)	.170	.173	n/a	.219	.238	.241
	Weibull($\beta = 2$)	.061	.059	.024	.064	.063	.060
	Weibull($\beta = 3.5$)	.049	.049	.042	.052	.053	.052
	Gamma($\beta = 2$)	.109	.101	.016	.180	.195	.207
	Normal(0,1)	.054	.052	.044	.053	.056	.053
	Uniforml(0,1)	.054	.062	.045	.063	.069	.069
	Beta(2,2)	.047	.049	.041	.050	.054	.052
	Beta(2,3)	.050	.052	.029	.053	.056	.053
15	Weibull($\beta = 1$)	.709	.755	n/a	.558	.651	.667
	Weibull($\beta = 2$)	.139	.139	.006	.102	.117	.117
	Weibull($\beta = 3.5$)	.050	.050	.055	.057	.058	.056
	Gamma($\beta = 2$)	.441	.451	.001	.423	.465	.476
	Normal(0,1)	.063	.059	.066	.063	.066	.061
	Uniforml(0,1)	.052	.099	.043	.086	.117	.131
	Beta(2,2)	.034	.044	.037	.044	.048	.050
	Beta(2,3)	.059	.065	.014	.061	.069	.065
25	Weibull($\beta = 1$)	.932	.957	n/a	.801	.881	.907
	Weibull($\beta = 2$)	.243	.250	n/a	.172	.176	.191
	Weibull($\beta = 3.5$)	.052	.053	n/a	.057	.058	.056
	Gamma($\beta = 2$)	.707	.733	n/a	.423	.465	.476
	Normal(0,1)	.069	.067	n/a	.076	.071	.073
	Uniforml(0,1)	.087	.201	n/a	.129	.172	.232
	Beta(2,2)	.036	.050	n/a	.067	.061	.073
	Beta(2,3)	.088	.102	n/a	.093	.094	.103

Validation section in the next section of this chapter. In this investigation, Coppa's critical values and power study were validated. Unfortunately, for some alternatives to the null, the Z^* statistic exhibits problems of bias and non-consistency, as noted in the literature review. Apparently, the gamma($\beta = 2$) distribution is one of these alternatives. At the 0.01 significance level, Z^* has zero power.

Table 4.20 Power Comparisons for H_0: Weibull($\beta = 3.5$), $\alpha = 0.01$

Sample size	Alternative Distribution	Test Statistics W_s^2	A_s^2	Z_e^*	KS	W^2	A^2
5	Weibull($\beta = 1$)	.060	.058	n/a	.106	.133	.152
	Weibull($\beta = 2$)	.013	.012	.006	.013	.016	.018
	Weibull($\beta = 3.5$)	.009	.010	.010	.008	.011	.013
	Gamma($\beta = 2$)	.028	.027	.003	.085	.109	.131
	Normal(0,1)	.011	.011	.009	.010	.011	.017
	Uniforml(0,1)	.011	.013	.011	.010	.014	.017
	Beta(2,2)	.010	.011	.008	.008	.010	.011
	Beta(2,3)	.011	.011	.006	.008	.010	.011
15	Weibull($\beta = 1$)	.504	.558	n/a	.372	.467	.467
	Weibull($\beta = 2$)	.044	.043	.001	.026	.033	.031
	Weibull($\beta = 3.5$)	.009	.010	.013	.013	.012	.010
	Gamma($\beta = 2$)	.238	.242	.000	.294	.343	.339
	Normal(0,1)	.014	.013	.020	.017	.016	.013
	Uniforml(0,1)	.009	.022	.008	.019	.023	.024
	Beta(2,2)	.005	.008	.006	.008	.010	.007
	Beta(2,3)	.011	.013	.002	.016	.015	.011
25	Weibull($\beta = 1$)	.838	.887	n/a	.600	.726	.782
	Weibull($\beta = 2$)	.100	.099	n/a	.054	.056	.071
	Weibull($\beta = 3.5$)	.011	.011	n/a	.010	.008	.009
	Gamma($\beta = 2$)	.512	.534	n/a	.434	.491	.520
	Normal(0,1)	.017	.015	n/a	.020	.020	.023
	Uniforml(0,1)	.017	.056	n/a	.030	.042	.067
	Beta(2,2)	.005	.009	n/a	.013	.014	.017
	Beta(2,3)	.022	.025	n/a	.022	.020	.025

4.6 Verification and Validation

In an effort to determine whether the Monte Carlo simulation models used in this thesis function properly and are an accurate representation of statistical theory, numerous verification and validation techniques were used throughout this research effort.

4.6.1 Verification. Verification is determining that a simulation computer code performs according to the intended logic (17:299). In other words, it answers the question, "was the model built right?" To assure an affirmative answer to that question for each model developed in this thesis, the computer programs were written and debugged in modules. Each program started as a moderately detailed model and was gradually made more complex to accommodate further objectives. However, before another level of complexity was added to the program, the execution of the previous level was tested rigorously. This included replicating the computer output with hand calculations when feasible.

4.6.2 Validation. Validation is concerned with determining whether the conceptual computer model is as accurate representation of the system under investigation (17:299). In other words, it answers the question, "was the right model built?" In this thesis, a valid model accurately represents what we would find from statistical theory. Fortunately, hypothesis tests and tests of fit afford some natural "cross checks". For instance, critical values of a given test statistic should increase as the significance level of the test decreases, regardless of sample size. This is true for all critical values generated in this thesis. Also, the power of a test should match the claimed level of significance when the null and alternative hypotheses are the same. For example, suppose we are testing the null hypotheses that a random sample of size 25 follows a Weibull distribution with shape parameter $\beta = 1$ at the $\alpha = 0.5$ significance level. If in our power study algorithm we draw numerous samples of size 25 from a Weibull($\beta = 1$) distribution as our alternative, we would expect 5 percent

of the samples to result in rejection of the null hypothesis. This result is evident throughout the power study in this thesis and is illustrated in Table 4.21 below. Finally, the curious behavior of the power of the Z^* statistic studied by Coppa (7)

Table 4.21 Example Power of W_s^2

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Weibull($\beta = 1$)	25	0.00	1.00	0.101	0.052	0.010
			0.00	0.80	0.102	0.051	0.011
			0.00	0.60	0.100	0.049	0.010
			0.20	1.00	0.102	0.050	0.010
			0.20	0.80	0.100	0.049	0.010
			0.40	1.00	0.099	0.049	0.009

was investigated and ultimately reproduced. Substituting the Z^* statistic into the code used for generating the critical values of A_s^2 and W_s^2, the critical values of Z^* were obtained. These values matched those presented in Coppa's thesis. Using these critical values in the power code developed for this thesis revealed the same power of the test achieved in Coppa's research. Not only does this help to validate the simulations employed in this research, it also validated the work accomplished by Coppa.

4.7 Summary

In this Chapter, the findings of this research effort were presented and discussed. Specifically, tabled critical values of the goodness-of-fit tests were shown and the use of these tables was demonstrated. The critical values of both W_s^2 and A_s^2 were found to have nice properties which, in general, allow for safe interpolation of the tabled values. The power of the tests were also shown for several alternatives to the following null distributions: Weibull($\beta = 1$), Weibull($\beta = 2$), and Weibull($\beta = 3.5$). For the most part, the power results were intuitive, including those involving censored samples. The power of the tests developed in thesis were also compared to

their prominent competitors. The comparison revealed that the new tests are very competitive, and in many cases the power of W_s^2 and A_s^2 is superior to the competition. Compared to each other, W_s^2 and A_s^2 demonstrated nearly equal power, although, for certain alternatives to the null distribution, A_s^2 was marginally superior. Finally, verification and validation techniques of the Monte Carlo simulations used to generate the results presented in this chapter were explained. The following chapter closes this thesis with conclusions and recommendations.

V. CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction

This chapter presents conclusions about the A_s^2 and W_s^2 goodness-of-fit tests developed in this thesis. Also discussed are recommendations for further research in this area.

5.2 Conclusions

Goodness-of-fit is a field of statistics that warrants attention, especially as more and more organizations in the Air Force, and in industry, are making decisions based on the outputs of computer simulations. Even in the hands of the best analyst, such outputs can be misleading if inputs are not modeled properly. No matter what the application, it is important for input models, typically theoretical probability distributions, to be consistent with the data collected through experimentation and observation. This thesis addresses assessing the fit of such data to the Weibull distribution. The Weibull distribution has many applications including use as a "failure model" in the field of reliability.

Most goodness-of-fit tests for the Weibull deal with the two-parameter form of the distribution (shape and scale unknown) in which the location parameter δ is assumed to be zero, or is estimated and transformed to zero. In the latter case, some data, and thus information about the underlying distribution, is lost. With location parameter $\delta = 0$, tests for the two-parameter Weibull are equivalent to tests for the extreme-value distribution when the raw data is log-transformed prior to testing. However, "fixing" the location parameter may limit the suitability of such tests in reliability studies. In reliability theory, the location parameter of the Weibull distribution indicates the value of a random variable X for which failures may begin to occur. If a particular component or system truly has a period of operation that is failure free or has the possibility of failing prior to operation, it is important for

the failure distribution to be flexible enough to model this. In many circumstances, tests for the Weibull with known shape (unknown location and scale) may be more appropriate, since there are various applications of the Weibull in which the shape parameter is known. As a result, two new procedures for testing the known-shape form of the Weibull distribution are presented in this document. These procedures involve EDF-type test statistics based on the normalized spacings of complete, and Type II censored data; a proven concept for the two-parameter form of the Weibull. Tests based on spacings eliminate the need for parameter estimation prior to the test and can be easily adapted for censored data. Of course, once a distribution is accepted as an adequate model, any unknown parameters must be estimated for use in a simulation.

Goodness-of-fit tests for the Weibull distribution with known shape are not new, nor are tests based on normalized spacings. In fact, a test for the known-shape form of the Weibull based on normalized spacings is currently available in the literature. However, the current test employs a test statistic which demonstrates problems of bias and non-consistency in a study of its power. These problems are eliminated in this research by employing the Anderson-Darling and Cramér-von Mises statistics based on normalized spacings, denoted by A_s^2 and W_s^2, respectively. In addition to this contribution, the following observations and conclusions are made from the critical value and power studies accomplished during this research effort:

1. The A_s^2 and W_s^2 test statistics are relatively easy to calculate since no parameter estimation is needed. However, computer accuracy appears to be a limitation in the evaluation of the expected values necessary to compute the test statistics. At sample sizes $n \geq 30$, equation (3.3), programmed in Matlab 4.0, yields values which diverge from the published values (see Harter (13)). For this reason, the published values should be used, when available. A table look-up procedure can be easily implemented.

2. Based on the verification and validation efforts described in Chapter 4, the tabled percentage points of the Anderson-Darling and Cramér-von Mises tests developed in this thesis are trustworthy.

3. For hypothesized shapes $\beta \geq 2$, the well-behaved nature of the critical values of both A_s^2 and W_s^2 allow for safe linear interpolation of the tabled values between sample sizes and between shapes.

4. With a few exceptions, the powers of A_s^2 and W_s^2 are nearly equal for a given hypothesized shape, alternative to the null, sample size, degree of censoring, and significance level.

5. For the uniform(0,1) and chi-square(1) alternatives, A_s^2 achieves considerably higher power than W_s^2 at sample sizes $n = 15$ and 25, regardless of the hypothesized Weibull shape β. For this reason, A_s^2 is the recommended test, overall.

6. Power comparisons of A_s^2 and W_s^2 to their prominent competitors (Z^*, KS, A^2, and W^2), for hypothesized shapes $\beta = 1$ and $\beta = 3.5$, show mixed results. When $\beta = 1$, A_s^2 and Z_s^2 achieve considerably higher power than KS, A^2, and W^2 for the given alternatives and sample sizes considered. Yet, Z^* achieves more power than all of the other tests. When $\beta = 3.5$, the power of Z^* essentially goes to zero, while A_s^2 and W_s^2 remain competitive with KS, A^2, and W^2.

5.3 Recommendations for Further Research

Since the A_s^2 and W_s^2 goodness-of-fit tests developed in this thesis are for small samples ($n \leq 40$), the next logical step is to investigate these tests for large samples. It would be especially useful to have tabled critical values for the asymptotic distributions of the test statistics. Of course, such an effort would require a new scheme to generate, or perhaps approximate, the expected values of the order statistics of large sample Weibull random variables for various known shapes.

Once the asymptotic percentage points are generated, critical value approximations can be considered. Successful modifications to the test statistics based on the asymptotic values, sample size and/or shape could reduce the number of tabled values required for the tests, thus simplifying the testing procedure.

Finally, more work can always be done with "small" samples. Critical values for censoring levels other than the ones used in this thesis can be generated. Along the same lines, power results can be benchmarked for more null shapes, more alternatives to the null, more samples sizes, and more degrees of censoring.

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Table A.1 A_s^2 Critical Values for Shape $\beta = 0.5$

	Sample	Cnsr level		Significance level α						
Shape	size	L	R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
0.5	5	0.00	1.00	1.779	2.057	2.423	2.966	3.944	4.932	6.284
0.5	5	0.00	0.80	1.592	1.841	2.170	2.638	3.483	4.366	5.593
0.5	5	0.00	0.60	1.437	1.677	1.990	2.447	3.230	4.024	5.063
0.5	5	0.20	1.00	1.531	1.766	2.079	2.521	3.324	4.142	5.242
0.5	5	0.20	0.80	1.374	1.592	1.884	2.296	3.012	3.742	4.676
0.5	5	0.40	1.00	1.384	1.610	1.897	2.318	3.018	3.732	4.650
0.5	10	0.00	1.00	2.108	2.443	2.888	3.544	4.739	6.018	7.711
0.5	10	0.00	0.80	1.870	2.167	2.556	3.132	4.143	5.215	6.636
0.5	10	0.00	0.60	1.699	1.960	2.298	2.794	3.694	4.659	6.006
0.5	10	0.20	1.00	1.705	1.964	2.313	2.821	3.733	4.699	5.969
0.5	10	0.20	0.80	1.490	1.711	2.006	2.439	3.226	4.017	5.067
0.5	10	0.40	1.00	1.509	1.734	2.039	2.484	3.245	4.048	5.112
0.5	15	0.00	1.00	2.290	2.659	3.152	3.871	5.153	6.547	8.376
0.5	15	0.00	0.80	2.040	2.369	2.798	3.431	4.571	5.771	7.365
0.5	15	0.00	0.60	1.838	2.123	2.505	3.066	4.088	5.156	6.614
0.5	15	0.20	1.00	1.802	2.077	2.442	2.983	3.973	5.035	6.523
0.5	15	0.20	0.80	1.555	1.781	2.094	2.551	3.371	4.207	5.388
0.5	15	0.40	1.00	1.572	1.801	2.116	2.573	3.419	4.296	5.503
0.5	20	0.00	1.00	2.399	2.781	3.311	4.066	5.493	6.982	8.959
0.5	20	0.00	0.80	2.160	2.514	2.993	3.677	4.888	6.169	7.894
0.5	20	0.00	0.60	1.965	2.275	2.693	3.312	4.385	5.507	7.024
0.5	20	0.20	1.00	1.843	2.126	2.511	3.084	4.081	5.143	6.644
0.5	20	0.20	0.80	1.578	1.811	2.115	2.570	3.384	4.257	5.417
0.5	20	0.40	1.00	1.615	1.850	2.164	2.637	3.483	4.363	5.604

Table A.2 A_s^2 Critical Values for Shape $\beta = 0.5$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
				Significance level α						
0.5	25	0.00	1.00	2.529	2.942	3.495	4.299	5.799	7.290	9.406
0.5	25	0.00	0.80	2.240	2.601	3.101	3.826	5.148	6.437	8.282
0.5	25	0.00	0.60	2.046	2.364	2.804	3.449	4.608	5.813	7.403
0.5	25	0.20	1.00	1.872	2.164	2.548	3.111	4.139	5.242	6.721
0.5	25	0.20	0.80	1.593	1.831	2.145	2.607	3.447	4.311	5.523
0.5	25	0.40	1.00	1.647	1.887	2.212	2.700	3.558	4.479	5.748
0.5	30	0.00	1.00	2.596	3.025	3.601	4.456	5.961	7.527	9.655
0.5	30	0.00	0.80	2.324	2.697	3.198	3.954	5.297	6.598	8.595
0.5	30	0.00	0.60	2.113	2.460	2.920	3.574	4.798	6.026	7.868
0.5	30	0.20	1.00	1.906	2.197	2.581	3.138	4.176	5.296	6.802
0.5	30	0.20	0.80	1.617	1.856	2.171	2.647	3.499	4.405	5.675
0.5	30	0.40	1.00	1.660	1.901	2.234	2.711	3.583	4.507	5.795
0.5	35	0.00	1.00	2.667	3.099	3.687	4.539	6.058	7.667	9.802
0.5	35	0.00	0.80	2.380	2.761	3.285	4.058	5.460	6.854	8.780
0.5	35	0.00	0.60	2.171	2.524	2.995	3.700	4.949	6.238	8.050
0.5	35	0.20	1.00	1.917	2.213	2.602	3.190	4.274	5.353	6.861
0.5	35	0.20	0.80	1.628	1.867	2.192	2.667	3.531	4.454	5.669
0.5	35	0.40	1.00	1.665	1.912	2.240	2.722	3.592	4.530	5.779
0.5	40	0.00	1.00	2.705	3.148	3.742	4.635	6.209	7.832	10.039
0.5	40	0.00	0.80	2.410	2.805	3.341	4.133	5.528	6.954	8.791
0.5	40	0.00	0.60	2.228	2.587	3.061	3.770	5.016	6.377	8.172
0.5	40	0.20	1.00	1.942	2.235	2.636	3.238	4.310	5.407	6.933
0.5	40	0.20	0.80	1.641	1.881	2.207	2.687	3.544	4.472	5.700
0.5	40	0.40	1.00	1.674	1.923	2.253	2.751	3.648	4.563	5.867

Table A.3 A_s^2 Critical Values for Shape $\beta = 1.0$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
				Significance level α						
1.0	5	0.00	1.00	1.281	1.458	1.695	2.047	2.659	3.314	4.208
1.0	5	0.00	0.80	1.275	1.459	1.702	2.058	2.698	3.338	4.196
1.0	5	0.00	0.60	1.230	1.426	1.693	2.071	2.738	3.404	4.323
1.0	5	0.20	1.00	1.238	1.414	1.655	1.995	2.587	3.198	4.057
1.0	5	0.20	0.80	1.219	1.411	1.673	2.047	2.710	3.382	4.308
1.0	5	0.40	1.00	1.212	1.409	1.672	2.056	2.726	3.395	4.286
1.0	10	0.00	1.00	1.248	1.410	1.627	1.953	2.523	3.112	3.939
1.0	10	0.00	0.80	1.243	1.407	1.626	1.946	2.519	3.122	3.950
1.0	10	0.00	0.60	1.242	1.405	1.627	1.961	2.550	3.168	4.013
1.0	10	0.20	1.00	1.240	1.402	1.627	1.938	2.523	3.111	3.909
1.0	10	0.20	0.80	1.234	1.405	1.626	1.954	2.547	3.164	4.026
1.0	10	0.40	1.00	1.250	1.419	1.634	1.961	2.533	3.137	3.966
1.0	15	0.00	1.00	1.249	1.412	1.624	1.941	2.492	3.080	3.900
1.0	15	0.00	0.80	1.254	1.415	1.630	1.953	2.524	3.107	3.926
1.0	15	0.00	0.60	1.249	1.411	1.628	1.951	2.508	3.088	3.906
1.0	15	0.20	1.00	1.245	1.407	1.623	1.932	2.490	3.081	3.882
1.0	15	0.20	0.80	1.251	1.415	1.638	1.962	2.539	3.169	3.964
1.0	15	0.40	1.00	1.247	1.407	1.627	1.953	2.534	3.139	3.984
1.0	20	0.00	1.00	1.241	1.402	1.616	1.925	2.489	3.093	3.909
1.0	20	0.00	0.80	1.240	1.402	1.618	1.928	2.482	3.066	3.899
1.0	20	0.00	0.60	1.253	1.414	1.628	1.944	2.510	3.140	3.966
1.0	20	0.20	1.00	1.251	1.413	1.633	1.948	2.507	3.098	3.912
1.0	20	0.20	0.80	1.250	1.414	1.626	1.944	2.513	3.130	3.973
1.0	20	0.40	1.00	1.243	1.407	1.623	1.932	2.494	3.098	3.911

Table A.4 A_s^2 Critical Values for Shape $\beta = 1.0$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
1.0	25	0.00	1.00	1.251	1.412	1.618	1.939	2.499	3.092	3.904
1.0	25	0.00	0.80	1.235	1.396	1.609	1.922	2.485	3.071	3.881
1.0	25	0.00	0.60	1.245	1.408	1.622	1.941	2.511	3.085	3.921
1.0	25	0.20	1.00	1.248	1.410	1.625	1.934	2.493	3.091	3.918
1.0	25	0.20	0.80	1.240	1.400	1.618	1.936	2.509	3.090	3.902
1.0	25	0.40	1.00	1.245	1.404	1.622	1.945	2.514	3.110	3.896
1.0	30	0.00	1.00	1.245	1.404	1.619	1.939	2.520	3.113	3.925
1.0	30	0.00	0.80	1.251	1.412	1.628	1.945	2.499	3.098	3.911
1.0	30	0.00	0.60	1.245	1.410	1.624	1.940	2.494	3.055	3.863
1.0	30	0.20	1.00	1.248	1.412	1.625	1.948	2.524	3.109	3.916
1.0	30	0.20	0.80	1.248	1.409	1.623	1.935	2.499	3.082	3.880
1.0	30	0.40	1.00	1.248	1.412	1.627	1.942	2.508	3.107	3.904
1.0	35	0.00	1.00	1.248	1.411	1.616	1.933	2.471	3.084	3.892
1.0	35	0.00	0.80	1.248	1.411	1.626	1.940	2.503	3.097	3.932
1.0	35	0.00	0.60	1.251	1.415	1.628	1.945	2.522	3.102	3.900
1.0	35	0.20	1.00	1.251	1.414	1.630	1.934	2.500	3.086	3.889
1.0	35	0.20	0.80	1.241	1.403	1.620	1.933	2.489	3.086	3.921
1.0	35	0.40	1.00	1.247	1.412	1.626	1.940	2.499	3.077	3.883
1.0	40	0.00	1.00	1.252	1.418	1.623	1.934	2.498	3.097	3.931
1.0	40	0.00	0.80	1.244	1.407	1.620	1.942	2.511	3.095	3.911
1.0	40	0.00	0.60	1.246	1.406	1.615	1.922	2.476	3.074	3.869
1.0	40	0.20	1.00	1.249	1.405	1.620	1.938	2.518	3.105	3.913
1.0	40	0.20	0.80	1.246	1.406	1.623	1.937	2.489	3.077	3.869
1.0	40	0.40	1.00	1.244	1.408	1.625	1.941	2.515	3.103	3.887

Table A.5 A_s^2 Critical Values for Shape $\beta = 1.5$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
				Significance level α						
1.5	5	0.00	1.00	1.141	1.289	1.492	1.785	2.329	2.885	3.639
1.5	5	0.00	0.80	1.162	1.325	1.547	1.875	2.452	3.078	3.908
1.5	5	0.00	0.60	1.166	1.352	1.608	1.988	2.658	3.332	4.280
1.5	5	0.20	1.00	1.170	1.336	1.556	1.874	2.455	3.038	3.868
1.5	5	0.20	0.80	1.184	1.377	1.631	2.007	2.657	3.339	4.263
1.5	5	0.40	1.00	1.179	1.367	1.614	1.989	2.647	3.323	4.230
1.5	10	0.00	1.00	1.101	1.233	1.412	1.665	2.127	2.631	3.306
1.5	10	0.00	0.80	1.142	1.288	1.481	1.757	2.258	2.814	3.539
1.5	10	0.00	0.60	1.169	1.319	1.521	1.818	2.351	2.923	3.717
1.5	10	0.20	1.00	1.160	1.305	1.492	1.777	2.281	2.845	3.602
1.5	10	0.20	0.80	1.199	1.359	1.567	1.882	2.439	3.022	3.827
1.5	10	0.40	1.00	1.185	1.339	1.543	1.848	2.387	2.955	3.719
1.5	15	0.00	1.00	1.070	1.198	1.363	1.612	2.052	2.536	3.196
1.5	15	0.00	0.80	1.119	1.256	1.437	1.710	2.193	2.697	3.412
1.5	15	0.00	0.60	1.146	1.292	1.483	1.765	2.277	2.817	3.505
1.5	15	0.20	1.00	1.145	1.288	1.477	1.757	2.242	2.757	3.456
1.5	15	0.20	0.80	1.196	1.350	1.559	1.865	2.405	2.949	3.701
1.5	15	0.40	1.00	1.181	1.333	1.537	1.828	2.359	2.895	3.604
1.5	20	0.00	1.00	1.056	1.182	1.350	1.594	2.034	2.493	3.154
1.5	20	0.00	0.80	1.100	1.235	1.412	1.678	2.160	2.640	3.314
1.5	20	0.00	0.60	1.137	1.277	1.464	1.738	2.219	2.741	3.425
1.5	20	0.20	1.00	1.142	1.285	1.477	1.750	2.244	2.774	3.465
1.5	20	0.20	0.80	1.190	1.345	1.549	1.852	2.378	2.924	3.712
1.5	20	0.40	1.00	1.172	1.318	1.518	1.800	2.327	2.861	3.617

Table A.6 A_s^2 Critical Values for Shape $\beta = 1.5$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
				Significance level α						
1.5	25	0.00	1.00	1.042	1.165	1.325	1.560	1.977	2.423	3.039
1.5	25	0.00	0.80	1.096	1.229	1.408	1.663	2.114	2.581	3.218
1.5	25	0.00	0.60	1.127	1.271	1.452	1.722	2.203	2.734	3.437
1.5	25	0.20	1.00	1.136	1.277	1.458	1.729	2.206	2.722	3.451
1.5	25	0.20	0.80	1.198	1.352	1.549	1.841	2.359	2.934	3.706
1.5	25	0.40	1.00	1.166	1.309	1.505	1.785	2.300	2.819	3.574
1.5	30	0.00	1.00	1.036	1.155	1.317	1.555	1.979	2.417	3.012
1.5	30	0.00	0.80	1.087	1.219	1.391	1.645	2.086	2.560	3.240
1.5	30	0.00	0.60	1.116	1.255	1.437	1.700	2.170	2.671	3.364
1.5	30	0.20	1.00	1.133	1.268	1.456	1.727	2.215	2.712	3.412
1.5	30	0.20	0.80	1.196	1.346	1.544	1.830	2.356	2.895	3.634
1.5	30	0.40	1.00	1.162	1.311	1.507	1.791	2.298	2.852	3.588
1.5	35	0.00	1.00	1.020	1.141	1.301	1.531	1.934	2.373	2.991
1.5	35	0.00	0.80	1.077	1.204	1.372	1.621	2.084	2.539	3.188
1.5	35	0.00	0.60	1.111	1.247	1.433	1.699	2.185	2.682	3.393
1.5	35	0.20	1.00	1.128	1.268	1.451	1.722	2.199	2.697	3.391
1.5	35	0.20	0.80	1.186	1.336	1.533	1.823	2.342	2.880	3.621
1.5	35	0.40	1.00	1.170	1.318	1.511	1.799	2.306	2.819	3.498
1.5	40	0.00	1.00	1.022	1.139	1.294	1.528	1.942	2.369	2.948
1.5	40	0.00	0.80	1.067	1.197	1.368	1.614	2.055	2.516	3.127
1.5	40	0.00	0.60	1.105	1.239	1.417	1.673	2.142	2.636	3.335
1.5	40	0.20	1.00	1.127	1.264	1.447	1.716	2.210	2.723	3.435
1.5	40	0.20	0.80	1.181	1.330	1.530	1.815	2.341	2.877	3.659
1.5	40	0.40	1.00	1.160	1.303	1.497	1.778	2.267	2.800	3.511

Table A.7 A_s^2 Critical Values for Shape $\beta = 2.0$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
2.0	5	0.00	1.00	1.105	1.245	1.444	1.733	2.260	2.827	3.569
2.0	5	0.00	0.80	1.144	1.308	1.530	1.844	2.424	3.004	3.833
2.0	5	0.00	0.60	1.162	1.354	1.605	1.983	2.648	3.325	4.273
2.0	5	0.20	1.00	1.150	1.315	1.531	1.847	2.407	3.000	3.827
2.0	5	0.20	0.80	1.183	1.377	1.635	2.014	2.672	3.345	4.230
2.0	5	0.40	1.00	1.164	1.350	1.599	1.966	2.622	3.317	4.191
2.0	10	0.00	1.00	1.058	1.183	1.350	1.597	2.050	2.539	3.213
2.0	10	0.00	0.80	1.107	1.246	1.428	1.704	2.188	2.702	3.439
2.0	10	0.00	0.60	1.149	1.300	1.497	1.783	2.310	2.865	3.657
2.0	10	0.20	1.00	1.126	1.265	1.455	1.734	2.229	2.774	3.510
2.0	10	0.20	0.80	1.195	1.352	1.561	1.873	2.430	2.977	3.746
2.0	10	0.40	1.00	1.171	1.322	1.528	1.825	2.371	2.944	3.730
2.0	15	0.00	1.00	1.019	1.138	1.292	1.524	1.932	2.379	2.956
2.0	15	0.00	0.80	1.088	1.219	1.396	1.648	2.103	2.589	3.250
2.0	15	0.00	0.60	1.124	1.266	1.451	1.720	2.218	2.743	3.476
2.0	15	0.20	1.00	1.113	1.253	1.437	1.709	2.185	2.680	3.352
2.0	15	0.20	0.80	1.181	1.334	1.531	1.826	2.371	2.914	3.665
2.0	15	0.40	1.00	1.161	1.307	1.502	1.794	2.313	2.860	3.617
2.0	20	0.00	1.00	1.003	1.117	1.266	1.485	1.874	2.303	2.888
2.0	20	0.00	0.80	1.069	1.195	1.364	1.613	2.061	2.529	3.192
2.0	20	0.00	0.60	1.113	1.249	1.426	1.691	2.167	2.676	3.320
2.0	20	0.20	1.00	1.108	1.244	1.422	1.694	2.171	2.670	3.276
2.0	20	0.20	0.80	1.176	1.323	1.525	1.815	2.324	2.882	3.649
2.0	20	0.40	1.00	1.145	1.285	1.476	1.753	2.243	2.784	3.493

(Columns 0.25–0.01: Significance level α)

Table A.8 A_s^2 Critical Values for Shape $\beta = 2.0$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
2.0	25	0.00	1.00	0.991	1.104	1.254	1.474	1.866	2.265	2.838
2.0	25	0.00	0.80	1.061	1.186	1.357	1.601	2.036	2.488	3.140
2.0	25	0.00	0.60	1.096	1.225	1.396	1.655	2.119	2.622	3.289
2.0	25	0.20	1.00	1.100	1.235	1.416	1.674	2.150	2.642	3.326
2.0	25	0.20	0.80	1.173	1.322	1.524	1.812	2.327	2.863	3.610
2.0	25	0.40	1.00	1.142	1.286	1.469	1.750	2.245	2.751	3.482
2.0	30	0.00	1.00	0.979	1.092	1.241	1.456	1.826	2.239	2.787
2.0	30	0.00	0.80	1.049	1.170	1.338	1.576	2.006	2.450	3.085
2.0	30	0.00	0.60	1.088	1.220	1.396	1.653	2.089	2.579	3.238
2.0	30	0.20	1.00	1.085	1.214	1.389	1.641	2.096	2.584	3.216
2.0	30	0.20	0.80	1.173	1.317	1.509	1.792	2.321	2.849	3.524
2.0	30	0.40	1.00	1.134	1.271	1.462	1.728	2.223	2.752	3.489
2.0	35	0.00	1.00	0.967	1.077	1.223	1.431	1.814	2.208	2.745
2.0	35	0.00	0.80	1.043	1.167	1.331	1.565	1.992	2.451	3.074
2.0	35	0.00	0.60	1.083	1.217	1.387	1.639	2.088	2.559	3.196
2.0	35	0.20	1.00	1.090	1.220	1.392	1.645	2.099	2.581	3.214
2.0	35	0.20	0.80	1.169	1.316	1.507	1.793	2.299	2.837	3.522
2.0	35	0.40	1.00	1.133	1.274	1.462	1.732	2.221	2.725	3.410
2.0	40	0.00	1.00	0.959	1.068	1.214	1.421	1.800	2.189	2.727
2.0	40	0.00	0.80	1.033	1.153	1.313	1.548	1.964	2.413	3.017
2.0	40	0.00	0.60	1.077	1.206	1.376	1.623	2.068	2.528	3.172
2.0	40	0.20	1.00	1.088	1.220	1.390	1.644	2.093	2.583	3.227
2.0	40	0.20	0.80	1.165	1.313	1.504	1.782	2.292	2.820	3.556
2.0	40	0.40	1.00	1.131	1.274	1.454	1.723	2.201	2.703	3.394

Table A.9 A_s^2 Critical Values for Shape $\beta = 2.5$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
2.5	5	0.00	1.00	1.100	1.241	1.438	1.733	2.248	2.796	3.562
2.5	5	0.00	0.80	1.142	1.305	1.521	1.833	2.400	2.985	3.798
2.5	5	0.00	0.60	1.159	1.346	1.602	1.980	2.625	3.290	4.196
2.5	5	0.20	1.00	1.143	1.309	1.524	1.838	2.411	3.012	3.857
2.5	5	0.20	0.80	1.167	1.353	1.604	1.979	2.652	3.331	4.244
2.5	5	0.40	1.00	1.162	1.352	1.597	1.967	2.628	3.308	4.231
2.5	10	0.00	1.00	1.041	1.166	1.328	1.569	2.011	2.478	3.127
2.5	10	0.00	0.80	1.103	1.241	1.426	1.703	2.185	2.688	3.381
2.5	10	0.00	0.60	1.148	1.297	1.493	1.785	2.318	2.882	3.645
2.5	10	0.20	1.00	1.120	1.260	1.442	1.717	2.218	2.718	3.438
2.5	10	0.20	0.80	1.182	1.337	1.546	1.846	2.398	2.967	3.751
2.5	10	0.40	1.00	1.155	1.305	1.501	1.804	2.343	2.894	3.678
2.5	15	0.00	1.00	1.006	1.124	1.279	1.509	1.921	2.360	2.931
2.5	15	0.00	0.80	1.085	1.214	1.386	1.637	2.103	2.575	3.245
2.5	15	0.00	0.60	1.128	1.266	1.449	1.723	2.211	2.753	3.466
2.5	15	0.20	1.00	1.106	1.241	1.421	1.675	2.149	2.640	3.321
2.5	15	0.20	0.80	1.177	1.330	1.529	1.824	2.354	2.913	3.618
2.5	15	0.40	1.00	1.145	1.286	1.478	1.760	2.288	2.811	3.554
2.5	20	0.00	1.00	0.989	1.105	1.255	1.468	1.855	2.265	2.845
2.5	20	0.00	0.80	1.061	1.190	1.358	1.604	2.056	2.511	3.117
2.5	20	0.00	0.60	1.112	1.248	1.427	1.685	2.148	2.641	3.355
2.5	20	0.20	1.00	1.088	1.222	1.396	1.648	2.100	2.582	3.205
2.5	20	0.20	0.80	1.167	1.315	1.508	1.800	2.317	2.852	3.574
2.5	20	0.40	1.00	1.143	1.286	1.473	1.749	2.248	2.776	3.458

Table A.10 A_s^2 Critical Values for Shape $\beta = 2.5$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
				Significance level α						
2.5	25	0.00	1.00	0.973	1.085	1.234	1.445	1.816	2.227	2.806
2.5	25	0.00	0.80	1.048	1.176	1.344	1.582	2.014	2.478	3.091
2.5	25	0.00	0.60	1.094	1.227	1.403	1.663	2.140	2.634	3.306
2.5	25	0.20	1.00	1.085	1.217	1.391	1.645	2.108	2.562	3.204
2.5	25	0.20	0.80	1.168	1.311	1.500	1.784	2.300	2.826	3.580
2.5	25	0.40	1.00	1.128	1.266	1.452	1.727	2.214	2.730	3.397
2.5	30	0.00	1.00	0.963	1.071	1.212	1.419	1.786	2.190	2.749
2.5	30	0.00	0.80	1.041	1.163	1.326	1.563	1.981	2.438	3.065
2.5	30	0.00	0.60	1.086	1.219	1.393	1.643	2.094	2.571	3.257
2.5	30	0.20	1.00	1.083	1.215	1.387	1.637	2.086	2.557	3.187
2.5	30	0.20	0.80	1.163	1.305	1.500	1.784	2.275	2.807	3.505
2.5	30	0.40	1.00	1.125	1.262	1.445	1.715	2.194	2.708	3.379
2.5	35	0.00	1.00	0.949	1.055	1.195	1.398	1.760	2.164	2.692
2.5	35	0.00	0.80	1.040	1.161	1.320	1.557	1.971	2.405	3.004
2.5	35	0.00	0.60	1.079	1.208	1.379	1.630	2.074	2.548	3.178
2.5	35	0.20	1.00	1.074	1.205	1.380	1.632	2.070	2.532	3.167
2.5	35	0.20	0.80	1.160	1.306	1.499	1.773	2.291	2.833	3.541
2.5	35	0.40	1.00	1.119	1.254	1.441	1.711	2.187	2.720	3.416
2.5	40	0.00	1.00	0.944	1.051	1.192	1.391	1.752	2.146	2.668
2.5	40	0.00	0.80	1.029	1.152	1.312	1.538	1.952	2.386	2.992
2.5	40	0.00	0.60	1.075	1.199	1.372	1.625	2.071	2.523	3.173
2.5	40	0.20	1.00	1.071	1.195	1.365	1.611	2.064	2.526	3.167
2.5	40	0.20	0.80	1.161	1.307	1.496	1.773	2.272	2.820	3.582
2.5	40	0.40	1.00	1.114	1.248	1.433	1.695	2.156	2.660	3.320

Table A.11 A_s^2 Critical Values for Shape $\beta = 3.0$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
3.0	5	0.00	1.00	1.089	1.235	1.421	1.702	2.220	2.750	3.492
3.0	5	0.00	0.80	1.142	1.300	1.521	1.838	2.423	3.024	3.862
3.0	5	0.00	0.60	1.154	1.342	1.594	1.960	2.634	3.318	4.214
3.0	5	0.20	1.00	1.147	1.308	1.521	1.840	2.403	2.988	3.821
3.0	5	0.20	0.80	1.174	1.362	1.615	1.979	2.652	3.309	4.216
3.0	5	0.40	1.00	1.165	1.357	1.606	1.973	2.610	3.304	4.214
3.0	10	0.00	1.00	1.035	1.159	1.324	1.566	1.997	2.449	3.066
3.0	10	0.00	0.80	1.107	1.244	1.428	1.694	2.182	2.700	3.369
3.0	10	0.00	0.60	1.156	1.300	1.494	1.782	2.315	2.865	3.623
3.0	10	0.20	1.00	1.115	1.252	1.439	1.717	2.199	2.726	3.411
3.0	10	0.20	0.80	1.180	1.334	1.542	1.844	2.385	2.948	3.732
3.0	10	0.40	1.00	1.154	1.304	1.502	1.804	2.321	2.857	3.627
3	15	0.00	1.00	1.007	1.123	1.272	1.498	1.913	2.343	2.930
3	15	0.00	0.80	1.085	1.210	1.385	1.632	2.097	2.581	3.246
3	15	0.00	0.60	1.125	1.266	1.451	1.719	2.211	2.734	3.420
3	15	0.20	1.00	1.095	1.225	1.404	1.663	2.125	2.620	3.294
3	15	0.20	0.80	1.174	1.325	1.522	1.814	2.347	2.877	3.614
3	15	0.40	1.00	1.139	1.280	1.468	1.738	2.235	2.753	3.477
3.0	20	0.00	1.00	0.993	1.107	1.258	1.470	1.860	2.256	2.805
3.0	20	0.00	0.80	1.066	1.194	1.364	1.605	2.059	2.530	3.144
3.0	20	0.00	0.60	1.117	1.255	1.433	1.700	2.178	2.682	3.352
3.0	20	0.20	1.00	1.086	1.216	1.396	1.650	2.102	2.593	3.251
3.0	20	0.20	0.80	1.167	1.313	1.507	1.791	2.303	2.840	3.552
3.0	20	0.40	1.00	1.124	1.267	1.453	1.729	2.224	2.749	3.434

Column headers 0.25–0.01: Significance level α

Table A.12 A_s^2 Critical Values for Shape $\beta = 3.0$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
3.0	25	0.00	1.00	0.973	1.084	1.228	1.444	1.833	2.231	2.763
3.0	25	0.00	0.80	1.054	1.178	1.341	1.583	2.012	2.476	3.093
3.0	25	0.00	0.60	1.094	1.225	1.399	1.657	2.129	2.611	3.291
3.0	25	0.20	1.00	1.082	1.212	1.384	1.633	2.090	2.560	3.247
3.0	25	0.20	0.80	1.167	1.314	1.510	1.788	2.295	2.827	3.565
3.0	25	0.40	1.00	1.121	1.259	1.446	1.717	2.194	2.696	3.398
3.0	30	0.00	1.00	0.960	1.072	1.214	1.418	1.793	2.198	2.748
3.0	30	0.00	0.80	1.041	1.164	1.331	1.568	1.988	2.435	3.055
3.0	30	0.00	0.60	1.090	1.218	1.388	1.635	2.098	2.576	3.235
3.0	30	0.20	1.00	1.075	1.203	1.373	1.620	2.052	2.516	3.181
3.0	30	0.20	0.80	1.165	1.313	1.505	1.785	2.307	2.841	3.528
3.0	30	0.40	1.00	1.115	1.253	1.432	1.690	2.170	2.668	3.382
3.0	35	0.00	1.00	0.951	1.058	1.197	1.397	1.763	2.149	2.686
3.0	35	0.00	0.80	1.040	1.162	1.324	1.558	1.980	2.412	3.044
3.0	35	0.00	0.60	1.090	1.221	1.393	1.642	2.090	2.549	3.171
3.0	35	0.20	1.00	1.073	1.199	1.369	1.614	2.056	2.516	3.149
3.0	35	0.20	0.80	1.164	1.304	1.491	1.772	2.269	2.806	3.513
3.0	35	0.40	1.00	1.115	1.250	1.432	1.701	2.169	2.660	3.321
3	40	0.00	1.00	0.945	1.047	1.187	1.387	1.751	2.114	2.634
3	40	0.00	0.80	1.031	1.152	1.318	1.547	1.966	2.404	2.987
3	40	0.00	0.60	1.079	1.207	1.380	1.629	2.074	2.550	3.210
3	40	0.20	1.00	1.067	1.196	1.365	1.612	2.054	2.506	3.126
3	40	0.20	0.80	1.161	1.307	1.506	1.789	2.280	2.814	3.527
3	40	0.40	1.00	1.112	1.246	1.426	1.693	2.169	2.652	3.312

Table A.13 A_s^2 Critical Values for Shape $\beta = 3.5$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
				Significance level α						
3.5	5	0.00	1.00	1.092	1.232	1.425	1.711	2.235	2.769	3.500
3.5	5	0.00	0.80	1.150	1.311	1.527	1.836	2.403	2.971	3.804
3.5	5	0.00	0.60	1.151	1.339	1.584	1.951	2.616	3.324	4.227
3.5	5	0.20	1.00	1.141	1.296	1.513	1.826	2.386	2.976	3.787
3.5	5	0.20	0.80	1.170	1.359	1.609	1.979	2.648	3.316	4.223
3.5	5	0.40	1.00	1.153	1.339	1.593	1.963	2.608	3.294	4.162
3.5	10	0.00	1.00	1.042	1.162	1.326	1.565	2.015	2.471	3.096
3.5	10	0.00	0.80	1.106	1.240	1.418	1.685	2.179	2.727	3.416
3.5	10	0.00	0.60	1.152	1.300	1.503	1.799	2.319	2.886	3.693
3.5	10	0.20	1.00	1.117	1.254	1.436	1.712	2.199	2.705	3.402
3.5	10	0.20	0.80	1.176	1.328	1.533	1.839	2.400	2.973	3.782
3.5	10	0.40	1.00	1.153	1.300	1.499	1.791	2.320	2.897	3.662
3.5	15	0.00	1.00	1.011	1.128	1.282	1.511	1.913	2.356	2.946
3.5	15	0.00	0.80	1.086	1.215	1.388	1.639	2.100	2.584	3.239
3.5	15	0.00	0.60	1.132	1.269	1.453	1.723	2.223	2.742	3.439
3.5	15	0.20	1.00	1.085	1.218	1.393	1.654	2.111	2.623	3.296
3.5	15	0.20	0.80	1.173	1.322	1.521	1.814	2.333	2.879	3.626
3.5	15	0.40	1.00	1.136	1.279	1.472	1.743	2.243	2.767	3.535
3.5	20	0.00	1.00	0.982	1.097	1.246	1.459	1.853	2.269	2.864
3.5	20	0.00	0.80	1.070	1.198	1.368	1.619	2.062	2.524	3.143
3.5	20	0.00	0.60	1.115	1.254	1.440	1.700	2.175	2.664	3.361
3.5	20	0.20	1.00	1.084	1.217	1.393	1.643	2.103	2.585	3.244
3.5	20	0.20	0.80	1.169	1.314	1.507	1.792	2.325	2.864	3.590
3.5	20	0.40	1.00	1.121	1.256	1.444	1.713	2.203	2.708	3.381

Table A.14 A_s^2 Critical Values for Shape $\beta = 3.5$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
3.5	25	0.00	1.00	0.970	1.077	1.222	1.433	1.807	2.191	2.732
3.5	25	0.00	0.80	1.056	1.182	1.344	1.586	2.023	2.453	3.060
3.5	25	0.00	0.60	1.107	1.241	1.416	1.677	2.147	2.634	3.312
3.5	25	0.20	1.00	1.073	1.204	1.374	1.621	2.072	2.539	3.205
3.5	25	0.20	0.80	1.165	1.312	1.504	1.788	2.285	2.816	3.537
3.5	25	0.40	1.00	1.118	1.255	1.437	1.706	2.182	2.671	3.376
3.5	30	0.00	1.00	0.965	1.073	1.212	1.420	1.789	2.186	2.752
3.5	30	0.00	0.80	1.053	1.175	1.337	1.578	2.006	2.451	3.090
3.5	30	0.00	0.60	1.093	1.223	1.398	1.658	2.120	2.616	3.267
3.5	30	0.20	1.00	1.064	1.192	1.359	1.604	2.040	2.504	3.136
3.5	30	0.20	0.80	1.159	1.310	1.500	1.785	2.288	2.818	3.552
3.5	30	0.40	1.00	1.110	1.246	1.427	1.686	2.155	2.636	3.310
3.5	35	0.00	1.00	0.952	1.060	1.200	1.407	1.781	2.156	2.733
3.5	35	0.00	0.80	1.043	1.169	1.329	1.566	1.991	2.432	3.032
3.5	35	0.00	0.60	1.086	1.215	1.383	1.631	2.091	2.569	3.244
3.5	35	0.20	1.00	1.069	1.196	1.366	1.605	2.036	2.503	3.155
3.5	35	0.20	0.80	1.166	1.312	1.507	1.785	2.283	2.813	3.535
3.5	35	0.40	1.00	1.106	1.240	1.417	1.680	2.152	2.640	3.331
3.5	40	0.00	1.00	0.949	1.054	1.198	1.397	1.766	2.149	2.677
3.5	40	0.00	0.80	1.032	1.153	1.314	1.552	1.981	2.421	3.048
3.5	40	0.00	0.60	1.085	1.214	1.382	1.637	2.086	2.560	3.249
3.5	40	0.20	1.00	1.064	1.194	1.363	1.603	2.038	2.503	3.160
3.5	40	0.20	0.80	1.157	1.303	1.494	1.776	2.284	2.810	3.543
3.5	40	0.40	1.00	1.107	1.244	1.417	1.673	2.148	2.629	3.287

Table A.15 A_s^2 Critical Values for Shape $\beta = 4.0$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
4.0	5	0.00	1.00	1.091	1.230	1.419	1.700	2.213	2.743	3.452
4.0	5	0.00	0.80	1.140	1.299	1.515	1.831	2.407	3.009	3.828
4.0	5	0.00	0.60	1.153	1.338	1.589	1.962	2.602	3.238	4.128
4.0	5	0.20	1.00	1.138	1.301	1.514	1.827	2.396	2.988	3.777
4.0	5	0.20	0.80	1.180	1.368	1.621	1.999	2.646	3.323	4.244
4.0	5	0.40	1.00	1.159	1.346	1.601	1.977	2.622	3.295	4.208
4.0	10	0.00	1.00	1.039	1.163	1.324	1.566	1.989	2.444	3.068
4.0	10	0.00	0.80	1.110	1.246	1.429	1.697	2.192	2.696	3.413
4.0	10	0.00	0.60	1.157	1.302	1.500	1.794	2.315	2.869	3.645
4.0	10	0.20	1.00	1.111	1.245	1.431	1.692	2.169	2.705	3.415
4.0	10	0.20	0.80	1.176	1.327	1.536	1.839	2.381	2.949	3.756
4.0	10	0.40	1.00	1.148	1.294	1.490	1.778	2.310	2.846	3.624
4.0	15	0.00	1.00	1.011	1.129	1.282	1.508	1.912	2.333	2.924
4.0	15	0.00	0.80	1.090	1.224	1.401	1.661	2.131	2.614	3.284
4.0	15	0.00	0.60	1.132	1.271	1.458	1.728	2.225	2.731	3.421
4.0	15	0.20	1.00	1.094	1.222	1.398	1.650	2.117	2.619	3.313
4.0	15	0.20	0.80	1.167	1.316	1.513	1.803	2.331	2.873	3.608
4.0	15	0.40	1.00	1.137	1.276	1.461	1.736	2.230	2.761	3.463
4.0	20	0.00	1.00	0.992	1.103	1.255	1.469	1.865	2.284	2.883
4.0	20	0.00	0.80	1.070	1.198	1.370	1.622	2.077	2.547	3.175
4.0	20	0.00	0.60	1.116	1.251	1.433	1.692	2.176	2.673	3.332
4.0	20	0.20	1.00	1.080	1.210	1.385	1.637	2.082	2.559	3.209
4.0	20	0.20	0.80	1.169	1.316	1.508	1.796	2.303	2.825	3.550
4.0	20	0.40	1.00	1.122	1.260	1.445	1.716	2.203	2.735	3.440

Table A.16 A_s^2 Critical Values for Shape $\beta = 4.0$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
4.0	25	0.00	1.00	0.977	1.087	1.236	1.445	1.825	2.233	2.800
4.0	25	0.00	0.80	1.070	1.196	1.364	1.612	2.043	2.516	3.163
4.0	25	0.00	0.60	1.114	1.250	1.430	1.692	2.151	2.646	3.329
4.0	25	0.20	1.00	1.071	1.199	1.373	1.619	2.056	2.526	3.151
4.0	25	0.20	0.80	1.162	1.304	1.498	1.781	2.285	2.838	3.568
4.0	25	0.40	1.00	1.114	1.250	1.433	1.698	2.175	2.688	3.379
4.0	30	0.00	1.00	0.966	1.073	1.220	1.427	1.800	2.176	2.749
4.0	30	0.00	0.80	1.054	1.178	1.342	1.578	2.007	2.473	3.106
4.0	30	0.00	0.60	1.103	1.232	1.405	1.664	2.123	2.603	3.257
4.0	30	0.20	1.00	1.067	1.198	1.366	1.609	2.057	2.503	3.118
4.0	30	0.20	0.80	1.158	1.302	1.490	1.769	2.280	2.789	3.526
4.0	30	0.40	1.00	1.114	1.248	1.431	1.689	2.165	2.666	3.366
4.0	35	0.00	1.00	0.959	1.069	1.212	1.420	1.783	2.157	2.715
4.0	35	0.00	0.80	1.056	1.180	1.342	1.582	2.013	2.451	3.076
4.0	35	0.00	0.60	1.094	1.227	1.402	1.661	2.118	2.599	3.286
4.0	35	0.20	1.00	1.058	1.182	1.357	1.599	2.044	2.513	3.155
4.0	35	0.20	0.80	1.158	1.303	1.495	1.775	2.270	2.789	3.475
4.0	35	0.40	1.00	1.100	1.234	1.409	1.670	2.129	2.635	3.294
4.0	40	0.00	1.00	0.949	1.058	1.202	1.407	1.772	2.155	2.661
4.0	40	0.00	0.80	1.043	1.163	1.326	1.565	1.976	2.411	3.017
4.0	40	0.00	0.60	1.094	1.226	1.404	1.652	2.109	2.598	3.250
4.0	40	0.20	1.00	1.060	1.185	1.355	1.601	2.034	2.472	3.127
4.0	40	0.20	0.80	1.162	1.307	1.498	1.774	2.276	2.794	3.530
4.0	40	0.40	1.00	1.101	1.237	1.415	1.674	2.137	2.619	3.270

Appendix B. Cramér-von Mises W_s^2 Critical Values

Table B.1 W_s^2 Critical Values for Shape $\beta = 0.5$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
0.5	5	0.00	1.00	0.281	0.323	0.387	0.476	0.613	0.719	0.820
0.5	5	0.00	0.80	0.266	0.308	0.359	0.423	0.505	0.563	0.611
0.5	5	0.00	0.60	0.246	0.265	0.283	0.302	0.319	0.327	0.331
0.5	5	0.20	1.00	0.260	0.301	0.352	0.413	0.497	0.555	0.602
0.5	5	0.20	0.80	0.240	0.258	0.277	0.296	0.315	0.325	0.330
0.5	5	0.40	1.00	0.241	0.260	0.278	0.297	0.315	0.325	0.330
0.5	10	0.00	1.00	0.343	0.402	0.479	0.593	0.790	0.980	1.226
0.5	10	0.00	0.80	0.305	0.357	0.425	0.521	0.681	0.836	1.030
0.5	10	0.00	0.60	0.277	0.322	0.378	0.458	0.592	0.730	0.881
0.5	10	0.20	1.00	0.287	0.333	0.395	0.485	0.644	0.801	0.990
0.5	10	0.20	0.80	0.252	0.291	0.343	0.415	0.539	0.662	0.812
0.5	10	0.40	1.00	0.256	0.296	0.349	0.423	0.551	0.676	0.828
0.5	15	0.00	1.00	0.379	0.445	0.532	0.662	0.891	1.113	1.418
0.5	15	0.00	0.80	0.337	0.396	0.473	0.581	0.773	0.965	1.206
0.5	15	0.00	0.60	0.300	0.350	0.417	0.514	0.677	0.839	1.043
0.5	15	0.20	1.00	0.306	0.358	0.427	0.527	0.707	0.894	1.135
0.5	15	0.20	0.80	0.263	0.305	0.364	0.447	0.591	0.733	0.923
0.5	15	0.40	1.00	0.267	0.310	0.367	0.453	0.603	0.758	0.951
0.5	20	0.00	1.00	0.399	0.469	0.565	0.707	0.958	1.212	1.558
0.5	20	0.00	0.80	0.360	0.421	0.507	0.632	0.844	1.055	1.332
0.5	20	0.00	0.60	0.324	0.381	0.453	0.560	0.743	0.924	1.159
0.5	20	0.20	1.00	0.314	0.368	0.442	0.552	0.742	0.928	1.194
0.5	20	0.20	0.80	0.270	0.313	0.372	0.455	0.610	0.761	0.961
0.5	20	0.40	1.00	0.275	0.320	0.381	0.471	0.632	0.790	1.005

Table B.2 W_s^2 Critical Values for Shape $\beta = 0.5$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
0.5	25	0.00	1.00	0.423	0.499	0.602	0.755	1.020	1.284	1.654
0.5	25	0.00	0.80	0.374	0.441	0.532	0.662	0.896	1.124	1.415
0.5	25	0.00	0.60	0.339	0.398	0.476	0.588	0.787	0.989	1.244
0.5	25	0.20	1.00	0.322	0.378	0.452	0.561	0.763	0.964	1.238
0.5	25	0.20	0.80	0.274	0.319	0.379	0.468	0.625	0.785	1.002
0.5	25	0.40	1.00	0.282	0.328	0.391	0.485	0.649	0.822	1.057
0.5	30	0.00	1.00	0.436	0.516	0.623	0.781	1.060	1.346	1.707
0.5	30	0.00	0.80	0.390	0.459	0.552	0.687	0.929	1.170	1.482
0.5	30	0.00	0.60	0.352	0.413	0.497	0.617	0.829	1.051	1.324
0.5	30	0.20	1.00	0.330	0.388	0.464	0.572	0.773	0.987	1.275
0.5	30	0.20	0.80	0.278	0.324	0.386	0.478	0.643	0.813	1.030
0.5	30	0.40	1.00	0.285	0.332	0.396	0.490	0.656	0.834	1.082
0.5	35	0.00	1.00	0.448	0.530	0.639	0.800	1.084	1.380	1.775
0.5	35	0.00	0.80	0.399	0.469	0.566	0.711	0.968	1.220	1.539
0.5	35	0.00	0.60	0.361	0.425	0.514	0.640	0.858	1.085	1.379
0.5	35	0.20	1.00	0.332	0.390	0.468	0.582	0.791	1.005	1.284
0.5	35	0.20	0.80	0.281	0.327	0.392	0.484	0.648	0.816	1.050
0.5	35	0.40	1.00	0.286	0.333	0.399	0.494	0.665	0.844	1.080
0.5	40	0.00	1.00	0.455	0.539	0.652	0.820	1.119	1.416	1.827
0.5	40	0.00	0.80	0.405	0.480	0.581	0.729	0.986	1.238	1.580
0.5	40	0.00	0.60	0.371	0.438	0.526	0.655	0.878	1.121	1.429
0.5	40	0.20	1.00	0.338	0.396	0.474	0.590	0.803	1.010	1.305
0.5	40	0.20	0.80	0.285	0.332	0.395	0.489	0.658	0.830	1.059
0.5	40	0.40	1.00	0.288	0.337	0.404	0.501	0.676	0.855	1.099

Table B.3 W_s^2 Critical Values for Shape $\beta = 1.0$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
1.0	5	0.00	1.00	0.227	0.257	0.298	0.360	0.463	0.558	0.668
1.0	5	0.00	0.80	0.228	0.262	0.303	0.358	0.437	0.501	0.556
1.0	5	0.00	0.60	0.226	0.245	0.266	0.287	0.310	0.321	0.328
1.0	5	0.20	1.00	0.216	0.249	0.289	0.344	0.424	0.487	0.549
1.0	5	0.20	0.80	0.225	0.244	0.264	0.286	0.309	0.321	0.328
1.0	5	0.40	1.00	0.224	0.243	0.264	0.286	0.309	0.321	0.328
1.0	10	0.00	1.00	0.211	0.243	0.284	0.346	0.451	0.562	0.702
1.0	10	0.00	0.80	0.211	0.242	0.283	0.344	0.449	0.554	0.695
1.0	10	0.00	0.60	0.212	0.243	0.283	0.341	0.444	0.545	0.672
1.0	10	0.20	1.00	0.211	0.241	0.282	0.342	0.448	0.554	0.690
1.0	10	0.20	0.80	0.211	0.242	0.283	0.343	0.444	0.542	0.675
1.0	10	0.40	1.00	0.213	0.244	0.284	0.341	0.442	0.538	0.667
1.0	15	0.00	1.00	0.210	0.242	0.284	0.344	0.454	0.569	0.719
1.0	15	0.00	0.80	0.212	0.244	0.285	0.346	0.455	0.568	0.721
1.0	15	0.00	0.60	0.212	0.243	0.283	0.343	0.449	0.554	0.692
1.0	15	0.20	1.00	0.210	0.242	0.283	0.343	0.451	0.562	0.714
1.0	15	0.20	0.80	0.212	0.243	0.286	0.347	0.455	0.563	0.701
1.0	15	0.40	1.00	0.211	0.243	0.285	0.347	0.452	0.564	0.703
1.0	20	0.00	1.00	0.209	0.240	0.282	0.344	0.453	0.568	0.728
1.0	20	0.00	0.80	0.209	0.240	0.282	0.342	0.451	0.565	0.724
1.0	20	0.00	0.60	0.211	0.242	0.284	0.346	0.455	0.569	0.720
1.0	20	0.20	1.00	0.211	0.243	0.285	0.348	0.459	0.572	0.724
1.0	20	0.20	0.80	0.211	0.243	0.285	0.346	0.454	0.570	0.718
1.0	20	0.40	1.00	0.210	0.241	0.283	0.344	0.452	0.565	0.713

Table B.4 W_s^2 Critical Values for Shape $\beta = 1.0$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
1.0	25	0.00	1.00	0.210	0.242	0.283	0.345	0.458	0.574	0.732
1.0	25	0.00	0.80	0.208	0.239	0.281	0.342	0.453	0.568	0.721
1.0	25	0.00	0.60	0.210	0.241	0.284	0.344	0.454	0.568	0.721
1.0	25	0.20	1.00	0.210	0.241	0.285	0.345	0.457	0.576	0.724
1.0	25	0.20	0.80	0.210	0.241	0.282	0.344	0.455	0.569	0.722
1.0	25	0.40	1.00	0.210	0.241	0.283	0.346	0.456	0.573	0.725
1.0	30	0.00	1.00	0.209	0.241	0.284	0.348	0.464	0.583	0.736
1.0	30	0.00	0.80	0.210	0.242	0.284	0.346	0.457	0.573	0.734
1.0	30	0.00	0.60	0.210	0.242	0.284	0.346	0.456	0.566	0.722
1.0	30	0.20	1.00	0.210	0.242	0.285	0.348	0.462	0.576	0.731
1.0	30	0.20	0.80	0.210	0.242	0.284	0.345	0.457	0.572	0.721
1.0	30	0.40	1.00	0.210	0.241	0.284	0.348	0.456	0.575	0.725
1.0	35	0.00	1.00	0.209	0.241	0.283	0.345	0.457	0.574	0.735
1.0	35	0.00	0.80	0.210	0.242	0.285	0.347	0.459	0.579	0.743
1.0	35	0.00	0.60	0.211	0.243	0.285	0.348	0.460	0.577	0.733
1.0	35	0.20	1.00	0.211	0.243	0.285	0.347	0.457	0.571	0.738
1.0	35	0.20	0.80	0.209	0.241	0.283	0.345	0.455	0.573	0.729
1.0	35	0.40	1.00	0.210	0.242	0.285	0.346	0.456	0.568	0.723
1.0	40	0.00	1.00	0.210	0.243	0.284	0.348	0.461	0.581	0.742
1.0	40	0.00	0.80	0.209	0.241	0.284	0.346	0.461	0.577	0.737
1.0	40	0.00	0.60	0.209	0.241	0.283	0.344	0.454	0.571	0.727
1.0	40	0.20	1.00	0.210	0.241	0.283	0.348	0.462	0.580	0.743
1.0	40	0.20	0.80	0.210	0.241	0.284	0.346	0.455	0.568	0.725
1.0	40	0.1.0	400	0.209	0.241	0.285	0.347	0.462	0.577	0.729

Table B.5 W_s^2 Critical Values for Shape $\beta = 1.5$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
1.5	5	0.00	1.00	0.198	0.224	0.257	0.308	0.400	0.491	0.593
1.5	5	0.00	0.80	0.204	0.234	0.272	0.325	0.407	0.473	0.538
1.5	5	0.00	0.60	0.219	0.238	0.260	0.283	0.308	0.320	0.328
1.5	5	0.20	1.00	0.205	0.235	0.274	0.327	0.408	0.472	0.538
1.5	5	0.20	0.80	0.221	0.240	0.261	0.284	0.308	0.320	0.328
1.5	5	0.40	1.00	0.220	0.240	0.260	0.283	0.307	0.320	0.328
1.5	10	0.00	1.00	0.185	0.211	0.245	0.294	0.381	0.475	0.598
1.5	10	0.00	0.80	0.194	0.222	0.257	0.310	0.404	0.501	0.627
1.5	10	0.00	0.60	0.200	0.229	0.266	0.320	0.413	0.504	0.630
1.5	10	0.20	1.00	0.196	0.224	0.261	0.312	0.405	0.504	0.637
1.5	10	0.20	0.80	0.204	0.234	0.272	0.328	0.426	0.521	0.644
1.5	10	0.40	1.00	0.202	0.231	0.269	0.322	0.417	0.512	0.632
1.5	15	0.00	1.00	0.179	0.204	0.237	0.284	0.373	0.462	0.586
1.5	15	0.00	0.80	0.188	0.215	0.250	0.302	0.394	0.492	0.621
1.5	15	0.00	0.60	0.194	0.222	0.259	0.312	0.407	0.503	0.631
1.5	15	0.20	1.00	0.192	0.219	0.256	0.309	0.403	0.500	0.625
1.5	15	0.20	0.80	0.203	0.232	0.271	0.327	0.429	0.528	0.656
1.5	15	0.40	1.00	0.200	0.229	0.267	0.324	0.421	0.519	0.646
1.5	20	0.00	1.00	0.176	0.200	0.233	0.281	0.367	0.458	0.576
1.5	20	0.00	0.80	0.185	0.211	0.246	0.297	0.389	0.483	0.614
1.5	20	0.00	0.60	0.192	0.219	0.255	0.307	0.401	0.499	0.627
1.5	20	0.20	1.00	0.192	0.219	0.256	0.310	0.404	0.507	0.641
1.5	20	0.20	0.80	0.200	0.230	0.270	0.327	0.428	0.531	0.670
1.5	20	0.40	1.00	0.196	0.225	0.264	0.318	0.417	0.518	0.649

Table B.6 W_s^2 Critical Values for Shape $\beta = 1.5$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
1.5	25	0.00	1.00	0.173	0.198	0.230	0.277	0.358	0.445	0.566
1.5	25	0.00	0.80	0.183	0.208	0.244	0.293	0.383	0.477	0.606
1.5	25	0.00	0.60	0.189	0.216	0.253	0.306	0.399	0.500	0.631
1.5	25	0.20	1.00	0.190	0.217	0.253	0.305	0.400	0.501	0.639
1.5	25	0.20	0.80	0.201	0.231	0.270	0.328	0.429	0.536	0.683
1.5	25	0.40	1.00	0.196	0.224	0.261	0.316	0.414	0.517	0.651
1.5	30	0.00	1.00	0.172	0.196	0.228	0.275	0.359	0.444	0.567
1.5	30	0.00	0.80	0.181	0.207	0.241	0.290	0.381	0.473	0.601
1.5	30	0.00	0.60	0.187	0.213	0.249	0.301	0.394	0.491	0.625
1.5	30	0.20	1.00	0.189	0.216	0.252	0.306	0.402	0.501	0.633
1.5	30	0.20	0.80	0.200	0.230	0.269	0.326	0.427	0.533	0.672
1.5	30	0.40	1.00	0.194	0.223	0.261	0.317	0.415	0.524	0.661
1.5	35	0.00	1.00	0.170	0.194	0.225	0.271	0.355	0.440	0.561
1.5	35	0.00	0.80	0.179	0.204	0.238	0.288	0.379	0.472	0.598
1.5	35	0.00	0.60	0.186	0.213	0.249	0.302	0.398	0.494	0.630
1.5	35	0.20	1.00	0.188	0.214	0.250	0.305	0.401	0.496	0.632
1.5	35	0.20	0.80	0.198	0.228	0.267	0.325	0.424	0.532	0.673
1.5	35	0.40	1.00	0.196	0.225	0.263	0.319	0.420	0.520	0.653
1.5	40	0.00	1.00	0.170	0.194	0.224	0.270	0.355	0.439	0.555
1.5	40	0.00	0.80	0.178	0.203	0.237	0.285	0.375	0.468	0.589
1.5	40	0.00	0.60	0.184	0.211	0.246	0.298	0.391	0.486	0.624
1.5	40	0.20	1.00	0.188	0.215	0.251	0.304	0.402	0.503	0.642
1.5	40	0.20	0.80	0.198	0.226	0.266	0.323	0.426	0.531	0.682
1.5	40	0.40	1.00	0.194	0.222	0.260	0.314	0.413	0.518	0.657

Table B.7 W_s^2 Critical Values for Shape $\beta = 2.0$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
2.0	5	0.00	1.00	0.193	0.218	0.250	0.298	0.390	0.477	0.582
2.0	5	0.00	0.80	0.201	0.231	0.268	0.321	0.401	0.466	0.533
2.0	5	0.00	0.60	0.218	0.238	0.259	0.283	0.307	0.320	0.328
2.0	5	0.20	1.00	0.202	0.231	0.269	0.321	0.401	0.468	0.532
2.0	5	0.20	0.80	0.221	0.241	0.262	0.284	0.308	0.320	0.328
2.0	5	0.40	1.00	0.218	0.238	0.259	0.282	0.307	0.320	0.328
2.0	10	0.00	1.00	0.178	0.202	0.233	0.281	0.367	0.452	0.575
2.0	10	0.00	0.80	0.187	0.214	0.249	0.300	0.390	0.482	0.604
2.0	10	0.00	0.60	0.197	0.225	0.262	0.313	0.404	0.494	0.615
2.0	10	0.20	1.00	0.190	0.218	0.253	0.304	0.396	0.489	0.618
2.0	10	0.20	0.80	0.202	0.232	0.272	0.328	0.422	0.518	0.639
2.0	10	0.40	1.00	0.200	0.229	0.266	0.320	0.414	0.507	0.631
2.0	15	0.00	1.00	0.170	0.193	0.223	0.267	0.348	0.429	0.541
2.0	15	0.00	0.80	0.183	0.208	0.242	0.291	0.378	0.468	0.589
2.0	15	0.00	0.60	0.190	0.217	0.252	0.304	0.396	0.491	0.619
2.0	15	0.20	1.00	0.186	0.213	0.248	0.300	0.392	0.484	0.607
2.0	15	0.20	0.80	0.200	0.228	0.266	0.321	0.422	0.525	0.656
2.0	15	0.40	1.00	0.196	0.224	0.261	0.315	0.413	0.510	0.633
2.0	20	0.00	1.00	0.166	0.188	0.218	0.261	0.339	0.420	0.529
2.0	20	0.00	0.80	0.179	0.203	0.236	0.286	0.371	0.459	0.584
2.0	20	0.00	0.60	0.187	0.213	0.248	0.298	0.390	0.484	0.611
2.0	20	0.20	1.00	0.185	0.211	0.246	0.298	0.389	0.486	0.610
2.0	20	0.20	0.80	0.198	0.226	0.264	0.321	0.419	0.521	0.663
2.0	20	0.40	1.00	0.192	0.220	0.256	0.309	0.404	0.500	0.633

Table B.8 W_s^2 Critical Values for Shape $\beta = 2.0$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
				Significance level α						
2.0	25	0.00	1.00	0.164	0.185	0.216	0.259	0.337	0.418	0.524
2.0	25	0.00	0.80	0.177	0.201	0.233	0.281	0.368	0.457	0.577
2.0	25	0.00	0.60	0.184	0.209	0.242	0.292	0.386	0.480	0.607
2.0	25	0.20	1.00	0.184	0.209	0.244	0.296	0.388	0.482	0.614
2.0	25	0.20	0.80	0.197	0.225	0.264	0.320	0.421	0.523	0.657
2.0	25	0.40	1.00	0.191	0.218	0.255	0.309	0.405	0.506	0.635
2.0	30	0.00	1.00	0.162	0.184	0.214	0.256	0.331	0.411	0.518
2.0	30	0.00	0.80	0.174	0.198	0.231	0.279	0.364	0.450	0.572
2.0	30	0.00	0.60	0.182	0.208	0.242	0.291	0.380	0.471	0.598
2.0	30	0.20	1.00	0.180	0.205	0.239	0.289	0.378	0.472	0.594
2.0	30	0.20	0.80	0.196	0.224	0.262	0.318	0.419	0.519	0.653
2.0	30	0.40	1.00	0.190	0.218	0.253	0.307	0.403	0.506	0.639
2.0	35	0.00	1.00	0.160	0.181	0.210	0.252	0.328	0.408	0.513
2.0	35	0.00	0.80	0.173	0.197	0.229	0.278	0.362	0.451	0.569
2.0	35	0.00	0.60	0.180	0.206	0.240	0.289	0.379	0.471	0.593
2.0	35	0.20	1.00	0.181	0.206	0.241	0.291	0.380	0.475	0.601
2.0	35	0.20	0.80	0.195	0.224	0.262	0.317	0.418	0.521	0.660
2.0	35	0.40	1.00	0.190	0.217	0.253	0.307	0.402	0.502	0.630
2.0	40	0.00	1.00	0.159	0.180	0.209	0.251	0.326	0.406	0.517
2.0	40	0.00	0.80	0.172	0.195	0.226	0.273	0.356	0.445	0.564
2.0	40	0.00	0.60	0.179	0.204	0.238	0.287	0.375	0.468	0.590
2.0	40	0.20	1.00	0.181	0.206	0.240	0.289	0.380	0.473	0.604
2.0	40	0.20	0.80	0.194	0.222	0.261	0.317	0.416	0.522	0.667
2.0	40	0.40	1.00	0.188	0.215	0.252	0.305	0.400	0.500	0.632

Table B.9 W_s^2 Critical Values for Shape $\beta = 2.5$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
				Significance level α						
2.5	5	0.00	1.00	0.192	0.217	0.250	0.298	0.389	0.474	0.578
2.5	5	0.00	0.80	0.201	0.229	0.268	0.319	0.399	0.466	0.534
2.5	5	0.00	0.60	0.218	0.238	0.259	0.283	0.307	0.320	0.328
2.5	5	0.20	1.00	0.202	0.230	0.269	0.320	0.403	0.470	0.537
2.5	5	0.20	0.80	0.219	0.238	0.259	0.282	0.307	0.320	0.328
2.5	5	0.40	1.00	0.218	0.238	0.259	0.282	0.307	0.320	0.328
2.5	10	0.00	1.00	0.175	0.198	0.230	0.276	0.358	0.441	0.554
2.5	10	0.00	0.80	0.187	0.213	0.247	0.298	0.388	0.473	0.592
2.5	10	0.00	0.60	0.197	0.225	0.260	0.312	0.403	0.495	0.621
2.5	10	0.20	1.00	0.190	0.216	0.250	0.300	0.390	0.482	0.605
2.5	10	0.20	0.80	0.202	0.230	0.268	0.323	0.417	0.510	0.633
2.5	10	0.40	1.00	0.198	0.225	0.261	0.314	0.409	0.498	0.619
2.5	15	0.00	1.00	0.167	0.190	0.220	0.264	0.344	0.424	0.534
2.5	15	0.00	0.80	0.181	0.206	0.240	0.288	0.375	0.466	0.589
2.5	15	0.00	0.60	0.191	0.217	0.253	0.305	0.395	0.489	0.615
2.5	15	0.20	1.00	0.184	0.210	0.245	0.295	0.385	0.477	0.602
2.5	15	0.20	0.80	0.198	0.227	0.265	0.321	0.420	0.523	0.647
2.5	15	0.40	1.00	0.193	0.220	0.256	0.309	0.405	0.500	0.632
2.5	20	0.00	1.00	0.164	0.186	0.215	0.258	0.333	0.412	0.523
2.5	20	0.00	0.80	0.177	0.201	0.235	0.282	0.368	0.460	0.577
2.5	20	0.00	0.60	0.187	0.213	0.248	0.297	0.388	0.480	0.610
2.5	20	0.20	1.00	0.181	0.206	0.241	0.290	0.378	0.472	0.595
2.5	20	0.20	0.80	0.196	0.225	0.262	0.317	0.417	0.519	0.654
2.5	20	0.40	1.00	0.192	0.219	0.255	0.308	0.404	0.498	0.625

Table B.10 W_s^2 Critical Values for Shape $\beta = 2.5$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
				Significance level α						
2.5	25	0.00	1.00	0.161	0.182	0.211	0.254	0.329	0.408	0.517
2.5	25	0.00	0.80	0.174	0.198	0.231	0.278	0.363	0.453	0.570
2.5	25	0.00	0.60	0.183	0.209	0.243	0.294	0.385	0.479	0.605
2.5	25	0.20	1.00	0.180	0.206	0.240	0.290	0.380	0.470	0.588
2.5	25	0.20	0.80	0.195	0.223	0.260	0.315	0.415	0.515	0.655
2.5	25	0.40	1.00	0.189	0.216	0.251	0.305	0.399	0.495	0.619
2.5	30	0.00	1.00	0.159	0.180	0.208	0.250	0.324	0.402	0.508
2.5	30	0.00	0.80	0.172	0.197	0.229	0.275	0.359	0.447	0.567
2.5	30	0.00	0.60	0.181	0.207	0.242	0.290	0.379	0.471	0.603
2.5	30	0.20	1.00	0.180	0.206	0.239	0.288	0.378	0.469	0.596
2.5	30	0.20	0.80	0.194	0.222	0.260	0.314	0.412	0.514	0.649
2.5	30	0.40	1.00	0.187	0.215	0.251	0.303	0.397	0.494	0.623
2.5	35	0.00	1.00	0.156	0.177	0.205	0.246	0.319	0.396	0.501
2.5	35	0.00	0.80	0.172	0.196	0.228	0.273	0.356	0.443	0.558
2.5	35	0.00	0.60	0.180	0.204	0.239	0.287	0.375	0.466	0.590
2.5	35	0.20	1.00	0.178	0.203	0.237	0.286	0.374	0.466	0.592
2.5	35	0.20	0.80	0.194	0.222	0.260	0.314	0.417	0.521	0.662
2.5	35	0.40	1.00	0.186	0.213	0.249	0.302	0.397	0.497	0.633
2.5	40	0.00	1.00	0.156	0.176	0.205	0.244	0.318	0.395	0.498
2.5	40	0.00	0.80	0.171	0.194	0.225	0.271	0.353	0.440	0.558
2.5	40	0.00	0.60	0.178	0.203	0.236	0.286	0.375	0.466	0.590
2.5	40	0.20	1.00	0.177	0.202	0.235	0.284	0.373	0.466	0.590
2.5	40	0.20	0.80	0.193	0.222	0.259	0.313	0.413	0.520	0.663
2.5	40	0.40	1.00	0.186	0.212	0.248	0.300	0.392	0.488	0.619

Table B.11 W_s^2 Critical Values for Shape $\beta = 3.0$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
				Significance level α						
3.0	5	0.00	1.00	0.191	0.216	0.247	0.293	0.378	0.467	0.574
3.0	5	0.00	0.80	0.200	0.229	0.266	0.319	0.402	0.469	0.535
3.0	5	0.00	0.60	0.217	0.237	0.259	0.281	0.307	0.320	0.328
3.0	5	0.20	1.00	0.201	0.231	0.269	0.320	0.400	0.466	0.534
3.0	5	0.20	0.80	0.220	0.239	0.260	0.283	0.307	0.320	0.328
3.0	5	0.40	1.00	0.219	0.239	0.259	0.282	0.306	0.320	0.328
3.0	10	0.00	1.00	0.174	0.197	0.228	0.274	0.354	0.439	0.550
3.0	10	0.00	0.80	0.187	0.213	0.247	0.298	0.386	0.477	0.594
3.0	10	0.00	0.60	0.197	0.226	0.261	0.313	0.404	0.495	0.614
3.0	10	0.20	1.00	0.189	0.215	0.250	0.300	0.389	0.479	0.603
3.0	10	0.20	0.80	0.202	0.230	0.268	0.323	0.416	0.509	0.630
3.0	10	0.40	1.00	0.196	0.224	0.261	0.313	0.404	0.497	0.619
3.0	15	0.00	1.00	0.167	0.190	0.218	0.262	0.341	0.422	0.532
3.0	15	0.00	0.80	0.181	0.206	0.240	0.289	0.375	0.466	0.586
3.0	15	0.00	0.60	0.191	0.217	0.252	0.303	0.394	0.489	0.611
3.0	15	0.20	1.00	0.183	0.209	0.242	0.292	0.381	0.471	0.596
3.0	15	0.20	0.80	0.198	0.227	0.264	0.318	0.417	0.516	0.646
3.0	15	0.40	1.00	0.192	0.219	0.255	0.306	0.398	0.494	0.619
3.0	20	0.00	1.00	0.164	0.186	0.216	0.258	0.334	0.411	0.513
3.0	20	0.00	0.80	0.177	0.202	0.235	0.283	0.369	0.460	0.577
3.0	20	0.00	0.60	0.187	0.214	0.249	0.299	0.392	0.487	0.606
3.0	20	0.20	1.00	0.181	0.206	0.240	0.290	0.379	0.470	0.595
3.0	20	0.20	0.80	0.196	0.224	0.261	0.317	0.414	0.514	0.648
3.0	20	0.40	1.00	0.188	0.215	0.252	0.304	0.397	0.497	0.625

Table B.12 W_s^2 Critical Values for Shape $\beta = 3.0$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
3.0	25	0.00	1.00	0.160	0.182	0.211	0.253	0.330	0.409	0.512
3.0	25	0.00	0.80	0.175	0.199	0.232	0.278	0.363	0.452	0.571
3.0	25	0.00	0.60	0.182	0.208	0.242	0.292	0.382	0.476	0.602
3.0	25	0.20	1.00	0.179	0.205	0.239	0.288	0.377	0.470	0.602
3.0	25	0.20	0.80	0.195	0.224	0.261	0.317	0.415	0.515	0.653
3.0	25	0.40	1.00	0.187	0.215	0.250	0.303	0.395	0.492	0.620
3.0	30	0.00	1.00	0.158	0.179	0.207	0.250	0.324	0.403	0.506
3.0	30	0.00	0.80	0.173	0.197	0.229	0.275	0.359	0.447	0.568
3.0	30	0.00	0.60	0.181	0.206	0.240	0.289	0.378	0.473	0.594
3.0	30	0.20	1.00	0.178	0.203	0.236	0.284	0.370	0.461	0.587
3.0	30	0.20	0.80	0.195	0.223	0.261	0.316	0.417	0.521	0.659
3.0	30	0.40	1.00	0.186	0.212	0.247	0.299	0.391	0.488	0.620
3.0	35	0.00	1.00	0.157	0.177	0.204	0.245	0.317	0.394	0.504
3.0	35	0.00	0.80	0.172	0.195	0.228	0.274	0.359	0.444	0.563
3.0	35	0.00	0.60	0.181	0.207	0.241	0.290	0.378	0.471	0.591
3.0	35	0.20	1.00	0.178	0.202	0.236	0.284	0.371	0.462	0.585
3.0	35	0.20	0.80	0.194	0.222	0.259	0.314	0.413	0.515	0.652
3.0	35	0.40	1.00	0.186	0.212	0.248	0.301	0.393	0.486	0.615
3.0	40	0.00	1.00	0.155	0.176	0.203	0.243	0.315	0.390	0.492
3.0	40	0.00	0.80	0.170	0.194	0.226	0.274	0.355	0.442	0.558
3.0	40	0.00	0.60	0.180	0.205	0.238	0.287	0.377	0.468	0.600
3.0	40	0.20	1.00	0.176	0.201	0.234	0.283	0.373	0.462	0.587
3.0	40	0.20	0.80	0.193	0.221	0.260	0.316	0.415	0.520	0.660
3.0	40	0.40	1.00	0.185	0.211	0.246	0.299	0.392	0.489	0.616

Table B.13 W_s^2 Critical Values for Shape $\beta = 3.5$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
3.5	5	0.00	1.00	0.192	0.216	0.247	0.294	0.382	0.470	0.573
3.5	5	0.00	0.80	0.201	0.231	0.268	0.321	0.399	0.466	0.530
3.5	5	0.00	0.60	0.217	0.237	0.258	0.281	0.306	0.320	0.328
3.5	5	0.20	1.00	0.200	0.229	0.267	0.318	0.397	0.463	0.531
3.5	5	0.20	0.80	0.219	0.239	0.260	0.282	0.307	0.320	0.328
3.5	5	0.40	1.00	0.217	0.237	0.259	0.282	0.306	0.320	0.328
3.5	10	0.00	1.00	0.175	0.198	0.230	0.276	0.358	0.443	0.552
3.5	10	0.00	0.80	0.188	0.213	0.247	0.296	0.386	0.479	0.603
3.5	10	0.00	0.60	0.197	0.225	0.262	0.315	0.405	0.498	0.618
3.5	10	0.20	1.00	0.189	0.215	0.249	0.299	0.386	0.474	0.595
3.5	10	0.20	0.80	0.201	0.230	0.267	0.321	0.415	0.512	0.632
3.5	10	0.40	1.00	0.197	0.224	0.261	0.313	0.405	0.497	0.617
3.5	15	0.00	1.00	0.168	0.191	0.221	0.264	0.342	0.426	0.539
3.5	15	0.00	0.80	0.182	0.207	0.240	0.288	0.376	0.468	0.583
3.5	15	0.00	0.60	0.191	0.218	0.253	0.304	0.398	0.491	0.614
3.5	15	0.20	1.00	0.182	0.207	0.240	0.290	0.378	0.467	0.592
3.5	15	0.20	0.80	0.198	0.226	0.264	0.320	0.416	0.515	0.652
3.5	15	0.40	1.00	0.192	0.219	0.255	0.307	0.399	0.496	0.627
3.5	20	0.00	1.00	0.163	0.184	0.213	0.256	0.331	0.413	0.524
3.5	20	0.00	0.80	0.179	0.203	0.236	0.285	0.370	0.458	0.575
3.5	20	0.00	0.60	0.187	0.214	0.249	0.300	0.391	0.485	0.613
3.5	20	0.20	1.00	0.180	0.206	0.239	0.289	0.377	0.470	0.592
3.5	20	0.20	0.80	0.196	0.224	0.262	0.317	0.418	0.519	0.654
3.5	20	0.40	1.00	0.188	0.214	0.250	0.303	0.394	0.489	0.613

(Significance level α spans columns 0.25 through 0.01.)

Table B.14 W_s^2 Critical Values for Shape $\beta = 3.5$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
3.5	25	0.00	1.00	0.160	0.181	0.209	0.251	0.326	0.403	0.503
3.5	25	0.00	0.80	0.175	0.199	0.231	0.280	0.363	0.449	0.566
3.5	25	0.00	0.60	0.185	0.211	0.245	0.296	0.387	0.482	0.607
3.5	25	0.20	1.00	0.178	0.203	0.237	0.285	0.373	0.464	0.588
3.5	25	0.20	0.80	0.195	0.223	0.261	0.317	0.412	0.514	0.653
3.5	25	0.40	1.00	0.187	0.213	0.249	0.300	0.393	0.486	0.613
3.5	30	0.00	1.00	0.159	0.180	0.208	0.249	0.323	0.398	0.506
3.5	30	0.00	0.80	0.174	0.198	0.230	0.277	0.363	0.449	0.568
3.5	30	0.00	0.60	0.182	0.207	0.241	0.292	0.381	0.477	0.606
3.5	30	0.20	1.00	0.176	0.201	0.234	0.282	0.368	0.456	0.582
3.5	30	0.20	0.80	0.194	0.222	0.260	0.315	0.413	0.516	0.658
3.5	30	0.40	1.00	0.185	0.211	0.247	0.298	0.389	0.479	0.611
3.5	35	0.00	1.00	0.157	0.178	0.206	0.247	0.321	0.398	0.505
3.5	35	0.00	0.80	0.172	0.196	0.229	0.276	0.361	0.449	0.566
3.5	35	0.00	0.60	0.180	0.206	0.239	0.289	0.378	0.474	0.599
3.5	35	0.20	1.00	0.177	0.202	0.235	0.282	0.369	0.461	0.589
3.5	35	0.20	0.80	0.194	0.222	0.260	0.315	0.416	0.517	0.656
3.5	35	0.40	1.00	0.184	0.211	0.246	0.297	0.389	0.487	0.621
3.5	40	0.00	1.00	0.156	0.178	0.205	0.246	0.320	0.397	0.505
3.5	40	0.00	0.80	0.171	0.194	0.226	0.274	0.358	0.446	0.569
3.5	40	0.00	0.60	0.180	0.206	0.240	0.289	0.378	0.470	0.599
3.5	40	0.20	1.00	0.177	0.201	0.235	0.284	0.371	0.461	0.589
3.5	40	0.20	0.80	0.193	0.221	0.258	0.314	0.413	0.518	0.658
3.5	40	0.40	1.00	0.184	0.210	0.245	0.296	0.389	0.483	0.613

Table B.15 W_s^2 Critical Values for Shape $\beta = 4.0$

Shape	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.25	0.20	0.15	0.10	0.05	0.025	0.01
4.0	5	0.00	1.00	0.191	0.215	0.246	0.293	0.380	0.465	0.567
4.0	5	0.00	0.80	0.199	0.229	0.267	0.319	0.400	0.467	0.533
4.0	5	0.00	0.60	0.217	0.237	0.258	0.282	0.306	0.319	0.327
4.0	5	0.20	1.00	0.200	0.229	0.267	0.319	0.400	0.467	0.534
4.0	5	0.20	0.80	0.220	0.240	0.261	0.283	0.307	0.320	0.328
4.0	5	0.40	1.00	0.218	0.238	0.259	0.282	0.307	0.320	0.328
4.0	10	0.00	1.00	0.174	0.198	0.229	0.274	0.355	0.439	0.545
4.0	10	0.00	0.80	0.188	0.214	0.249	0.299	0.389	0.477	0.594
4.0	10	0.00	0.60	0.197	0.225	0.262	0.314	0.403	0.496	0.614
4.0	10	0.20	1.00	0.188	0.214	0.248	0.299	0.387	0.477	0.600
4.0	10	0.20	0.80	0.200	0.229	0.266	0.321	0.415	0.508	0.630
4.0	10	0.40	1.00	0.196	0.224	0.260	0.311	0.401	0.493	0.612
4.0	15	0.00	1.00	0.168	0.190	0.221	0.265	0.342	0.419	0.528
4.0	15	0.00	0.80	0.183	0.208	0.242	0.291	0.380	0.472	0.592
4.0	15	0.00	0.60	0.191	0.218	0.253	0.305	0.395	0.489	0.616
4.0	15	0.20	1.00	0.183	0.208	0.242	0.291	0.380	0.473	0.596
4.0	15	0.20	0.80	0.197	0.225	0.263	0.318	0.415	0.518	0.642
4.0	15	0.40	1.00	0.192	0.219	0.254	0.306	0.398	0.490	0.614
4.0	20	0.00	1.00	0.164	0.186	0.215	0.258	0.335	0.415	0.528
4.0	20	0.00	0.80	0.178	0.203	0.236	0.285	0.373	0.462	0.588
4.0	20	0.00	0.60	0.187	0.214	0.247	0.298	0.390	0.485	0.608
4.0	20	0.20	1.00	0.180	0.205	0.238	0.288	0.376	0.465	0.588
4.0	20	0.20	0.80	0.196	0.224	0.262	0.315	0.416	0.509	0.645
4.0	20	0.40	1.00	0.188	0.215	0.251	0.302	0.394	0.494	0.629

Table B.16 W_s^2 Critical Values for Shape $\beta = 4.0$

Shape	Sample size	Cnsr level L	Cnsr level R	0.25	0.20	0.15	0.10	0.05	0.025	0.01
				Significance level α						
4.0	25	0.00	1.00	0.161	0.183	0.212	0.254	0.329	0.408	0.521
4.0	25	0.00	0.80	0.178	0.202	0.235	0.283	0.370	0.460	0.582
4.0	25	0.00	0.60	0.186	0.212	0.247	0.297	0.387	0.481	0.610
4.0	25	0.20	1.00	0.178	0.203	0.237	0.285	0.370	0.461	0.583
4.0	25	0.20	0.80	0.194	0.222	0.259	0.315	0.413	0.513	0.656
4.0	25	0.40	1.00	0.186	0.212	0.247	0.300	0.391	0.488	0.616
4.0	30	0.00	1.00	0.159	0.180	0.208	0.251	0.325	0.402	0.511
4.0	30	0.00	0.80	0.175	0.199	0.231	0.278	0.363	0.453	0.575
4.0	30	0.00	0.60	0.183	0.209	0.243	0.295	0.383	0.477	0.602
4.0	30	0.20	1.00	0.176	0.202	0.235	0.283	0.370	0.457	0.579
4.0	30	0.20	0.80	0.194	0.221	0.258	0.312	0.410	0.510	0.652
4.0	30	0.40	1.00	0.186	0.212	0.247	0.298	0.389	0.488	0.624
4.0	35	0.00	1.00	0.158	0.179	0.208	0.249	0.321	0.394	0.506
4.0	35	0.00	0.80	0.175	0.200	0.231	0.278	0.363	0.450	0.570
4.0	35	0.00	0.60	0.182	0.207	0.242	0.293	0.384	0.479	0.604
4.0	35	0.20	1.00	0.175	0.199	0.233	0.281	0.369	0.461	0.586
4.0	35	0.20	0.80	0.193	0.221	0.258	0.313	0.411	0.514	0.646
4.0	35	0.40	1.00	0.183	0.209	0.244	0.295	0.387	0.485	0.610
4.0	40	0.00	1.00	0.156	0.177	0.206	0.248	0.320	0.397	0.498
4.0	40	0.00	0.80	0.172	0.196	0.228	0.275	0.359	0.446	0.567
4.0	40	0.00	0.60	0.182	0.208	0.243	0.293	0.381	0.477	0.605
4.0	40	0.20	1.00	0.175	0.199	0.232	0.281	0.368	0.457	0.582
4.0	40	0.20	0.80	0.193	0.222	0.259	0.315	0.415	0.515	0.657
4.0	40	0.40	1.00	0.183	0.209	0.244	0.295	0.387	0.480	0.608

Appendix C. Power Study of A_s^2, H_0: Weibull(β = 1)

Table C.1 Power of the Test: A_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Weibull($\beta = 1$)	5	0.00	1.00	0.100	0.051	0.010
			0.00	0.80	0.102	0.050	0.010
			0.00	0.60	0.099	0.051	0.010
			0.20	1.00	0.100	0.051	0.010
			0.20	0.80	0.101	0.050	0.010
			0.40	1.00	0.098	0.049	0.011
Weibull($\beta = 1$)	Weibull($\beta = 2$)	5	0.00	1.00	0.169	0.095	0.021
			0.00	0.80	0.127	0.065	0.015
			0.00	0.60	0.105	0.053	0.011
			0.20	1.00	0.101	0.051	0.010
			0.20	0.80	0.098	0.050	0.010
			0.40	1.00	0.097	0.047	0.010
Weibull($\beta = 1$)	Weibull($\beta = 3.5$)	5	0.00	1.00	0.256	0.154	0.038
			0.00	0.80	0.165	0.089	0.020
			0.00	0.60	0.116	0.059	0.012
			0.20	1.00	0.118	0.061	0.012
			0.20	0.80	0.101	0.051	0.010
			0.40	1.00	0.096	0.047	0.010
Weibull($\beta = 1$)	Gamma($\beta = 2$)	5	0.00	1.00	0.116	0.060	0.012
			0.00	0.80	0.106	0.053	0.010
			0.00	0.60	0.098	0.048	0.009
			0.20	1.00	0.094	0.047	0.009
			0.20	0.80	0.098	0.049	0.009
			0.40	1.00	0.097	0.048	0.010
Weibull($\beta = 1$)	Normal(0,1)	5	0.00	1.00	0.256	0.159	0.042
			0.00	0.80	0.170	0.093	0.022
			0.00	0.60	0.116	0.059	0.013
			0.20	1.00	0.118	0.061	0.013
			0.20	0.80	0.099	0.049	0.011
			0.40	1.00	0.096	0.048	0.009

Table C.2 Power of the Test: A_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Uniform(0,1)	5	0.00	1.00	0.238	0.146	0.041
			0.00	0.80	0.140	0.074	0.017
			0.00	0.60	0.109	0.055	0.010
			0.20	1.00	0.157	0.088	0.020
			0.20	0.80	0.106	0.054	0.012
			0.40	1.00	0.119	0.061	0.013
Weibull($\beta = 1$)	Beta(2,2)	5	0.00	1.00	0.241	0.147	0.036
			0.00	0.80	0.152	0.082	0.019
			0.00	0.60	0.110	0.054	0.011
			0.20	1.00	0.128	0.067	0.014
			0.20	0.80	0.104	0.053	0.011
			0.40	1.00	0.103	0.051	0.010
Weibull($\beta = 1$)	Beta(2,3)	5	0.00	1.00	0.192	0.109	0.025
			0.00	0.80	0.134	0.069	0.015
			0.00	0.60	0.106	0.053	0.010
			0.20	1.00	0.110	0.057	0.012
			0.20	0.80	0.099	0.050	0.009
			0.40	1.00	0.099	0.049	0.010
Weibull($\beta = 1$)	Chi-square(1)	5	0.00	1.00	0.176	0.106	0.034
			0.00	0.80	0.154	0.088	0.026
			0.00	0.60	0.131	0.071	0.017
			0.20	1.00	0.139	0.077	0.020
			0.20	0.80	0.117	0.061	0.012
			0.40	1.00	0.111	0.057	0.013
Weibull($\beta = 1$)	Chi-square(4)	5	0.00	1.00	0.119	0.061	0.013
			0.00	0.80	0.109	0.055	0.011
			0.00	0.60	0.100	0.050	0.010
			0.20	1.00	0.094	0.048	0.009
			0.20	0.80	0.096	0.048	0.010
			0.40	1.00	0.098	0.049	0.010

Table C.3 Power of the Test: A_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Log-normal(0,1)	5	0.00	1.00	0.121	0.065	0.015
			0.00	0.80	0.098	0.048	0.010
			0.00	0.60	0.095	0.048	0.010
			0.20	1.00	0.123	0.068	0.016
			0.20	0.80	0.101	0.051	0.010
			0.40	1.00	0.110	0.055	0.011
Weibull($\beta = 1$)	Log-logistic(0,1)	5	0.00	1.00	0.301	0.223	0.117
			0.00	0.80	0.146	0.084	0.023
			0.00	0.60	0.105	0.053	0.010
			0.20	1.00	0.256	0.180	0.081
			0.20	0.80	0.125	0.065	0.014
			0.40	1.00	0.177	0.104	0.030
Weibull($\beta = 1$)	Log-double exp(0,1)	5	0.00	1.00	0.232	0.164	0.078
			0.00	0.80	0.121	0.063	0.015
			0.00	0.60	0.101	0.052	0.011
			0.20	1.00	0.220	0.149	0.066
			0.20	0.80	0.108	0.054	0.012
			0.40	1.00	0.169	0.097	0.026
Weibull($\beta = 1$)	Log-Cauchy(0,1)	5	0.00	1.00	0.499	0.442	0.360
			0.00	0.80	0.215	0.148	0.079
			0.00	0.60	0.139	0.083	0.031
			0.20	1.00	0.460	0.403	0.317
			0.20	0.80	0.162	0.101	0.039
			0.40	1.00	0.380	0.314	0.224

Table C.4 Power of the Test: A_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Weibull($\beta = 1$)	15	0.00	1.00	0.101	0.051	0.010
			0.00	0.80	0.099	0.049	0.010
			0.00	0.60	0.100	0.050	0.011
			0.20	1.00	0.100	0.051	0.010
			0.20	0.80	0.098	0.049	0.010
			0.40	1.00	0.099	0.051	0.011
Weibull($\beta = 1$)	Weibull($\beta = 2$)	15	0.00	1.00	0.561	0.427	0.194
			0.00	0.80	0.366	0.250	0.091
			0.00	0.60	0.237	0.146	0.043
			0.20	1.00	0.239	0.150	0.044
			0.20	0.80	0.145	0.080	0.019
			0.40	1.00	0.141	0.076	0.017
Weibull($\beta = 1$)	Weibull($\beta = 3.5$)	15	0.00	1.00	0.840	0.757	0.527
			0.00	0.80	0.625	0.503	0.269
			0.00	0.60	0.410	0.295	0.119
			0.20	1.00	0.403	0.280	0.107
			0.20	0.80	0.211	0.126	0.036
			0.40	1.00	0.195	0.114	0.030
Weibull($\beta = 1$)	Gamma($\beta = 2$)	15	0.00	1.00	0.228	0.143	0.041
			0.00	0.80	0.180	0.105	0.028
			0.00	0.60	0.146	0.082	0.021
			0.20	1.00	0.124	0.067	0.016
			0.20	0.80	0.105	0.054	0.011
			0.40	1.00	0.098	0.049	0.010
Weibull($\beta = 1$)	Normal(0,1)	15	0.00	1.00	0.842	0.765	0.551
			0.00	0.80	0.661	0.547	0.317
			0.00	0.60	0.458	0.341	0.151
			0.20	1.00	0.381	0.263	0.098
			0.20	0.80	0.204	0.122	0.035
			0.40	1.00	0.179	0.101	0.026

Table C.5 Power of the Test: A_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Uniform(0,1)	15	0.00	1.00	0.714	0.591	0.325
			0.00	0.80	0.297	0.194	0.068
			0.00	0.60	0.151	0.086	0.022
			0.20	1.00	0.591	0.457	0.219
			0.20	0.80	0.202	0.122	0.034
			0.40	1.00	0.429	0.299	0.112
Weibull($\beta = 1$)	Beta(2,2)	15	0.00	1.00	0.801	0.695	0.422
			0.00	0.80	0.480	0.353	0.150
			0.00	0.60	0.272	0.175	0.057
			0.20	1.00	0.477	0.345	0.138
			0.20	0.80	0.199	0.118	0.033
			0.40	1.00	0.278	0.175	0.050
Weibull($\beta = 1$)	Beta(2,3)	15	0.00	1.00	0.649	0.517	0.255
			0.00	0.80	0.379	0.260	0.095
			0.00	0.60	0.232	0.145	0.043
			0.20	1.00	0.338	0.224	0.075
			0.20	0.80	0.162	0.093	0.023
			0.40	1.00	0.194	0.112	0.029
Weibull($\beta = 1$)	Chi-square(1)	15	0.00	1.00	0.492	0.389	0.210
			0.00	0.80	0.417	0.314	0.156
			0.00	0.60	0.338	0.244	0.110
			0.20	1.00	0.245	0.164	0.061
			0.20	0.80	0.184	0.114	0.036
			0.40	1.00	0.150	0.086	0.022
Weibull($\beta = 1$)	Chi-square(4)	15	0.00	1.00	0.231	0.144	0.043
			0.00	0.80	0.182	0.106	0.028
			0.00	0.60	0.146	0.082	0.021
			0.20	1.00	0.122	0.067	0.016
			0.20	0.80	0.105	0.054	0.011
			0.40	1.00	0.099	0.051	0.011

Table C.6 Power of the Test: A_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Log-normal(0,1)	15	0.00	1.00	0.210	0.139	0.056
			0.00	0.80	0.106	0.054	0.012
			0.00	0.60	0.097	0.049	0.010
			0.20	1.00	0.223	0.150	0.060
			0.20	0.80	0.113	0.057	0.013
			0.40	1.00	0.192	0.122	0.043
Weibull($\beta = 1$)	Log-logistic(0,1)	15	0.00	1.00	0.763	0.705	0.578
			0.00	0.80	0.373	0.279	0.144
			0.00	0.60	0.181	0.112	0.035
			0.20	1.00	0.675	0.607	0.473
			0.20	0.80	0.261	0.178	0.073
			0.40	1.00	0.557	0.478	0.339
Weibull($\beta = 1$)	Log-double exp(0,1)	15	0.00	1.00	0.552	0.481	0.364
			0.00	0.80	0.173	0.105	0.034
			0.00	0.60	0.124	0.068	0.016
			0.20	1.00	0.567	0.498	0.372
			0.20	0.80	0.163	0.099	0.032
			0.40	1.00	0.529	0.450	0.316
Weibull($\beta = 1$)	Log-Cauchy(0,1)	15	0.00	1.00	0.898	0.877	0.835
			0.00	0.80	0.423	0.342	0.226
			0.00	0.60	0.245	0.173	0.083
			0.20	1.00	0.872	0.849	0.806
			0.20	0.80	0.318	0.245	0.150
			0.40	1.00	0.857	0.830	0.775

Table C.7 Power of the Test: A_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Weibull($\beta = 1$)	25	0.00	1.00	0.100	0.052	0.010
			0.00	0.80	0.102	0.051	0.010
			0.00	0.60	0.100	0.049	0.010
			0.20	1.00	0.101	0.052	0.010
			0.20	0.80	0.099	0.049	0.010
			0.40	1.00	0.098	0.049	0.010
Weibull($\beta = 1$)	Weibull($\beta = 2$)	25	0.00	1.00	0.844	0.755	0.517
			0.00	0.80	0.645	0.522	0.273
			0.00	0.60	0.436	0.310	0.128
			0.20	1.00	0.400	0.280	0.102
			0.20	0.80	0.213	0.130	0.040
			0.40	1.00	0.203	0.122	0.033
Weibull($\beta = 1$)	Weibull($\beta = 3.5$)	25	0.00	1.00	0.984	0.968	0.895
			0.00	0.80	0.900	0.838	0.661
			0.00	0.60	0.715	0.608	0.365
			0.20	1.00	0.658	0.531	0.271
			0.20	0.80	0.345	0.233	0.085
			0.40	1.00	0.322	0.212	0.072
Weibull($\beta = 1$)	Gamma($\beta = 2$)	25	0.00	1.00	0.400	0.281	0.104
			0.00	0.80	0.315	0.207	0.069
			0.00	0.60	0.234	0.145	0.043
			0.20	1.00	0.154	0.088	0.022
			0.20	0.80	0.124	0.065	0.016
			0.40	1.00	0.113	0.059	0.013
Weibull($\beta = 1$)	Normal(0,1)	25	0.00	1.00	0.984	0.970	0.904
			0.00	0.80	0.915	0.865	0.708
			0.00	0.60	0.762	0.668	0.448
			0.20	1.00	0.618	0.492	0.243
			0.20	0.80	0.337	0.229	0.085
			0.40	1.00	0.285	0.182	0.060

Table C.8 Power of the Test: A_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Uniform(0,1)	25	0.00	1.00	0.943	0.888	0.693
			0.00	0.80	0.483	0.359	0.157
			0.00	0.60	0.208	0.127	0.037
			0.20	1.00	0.865	0.774	0.518
			0.20	0.80	0.319	0.211	0.077
			0.40	1.00	0.720	0.594	0.329
Weibull($\beta = 1$)	Beta(2,2)	25	0.00	1.00	0.980	0.955	0.842
			0.00	0.80	0.778	0.673	0.417
			0.00	0.60	0.491	0.365	0.160
			0.20	1.00	0.756	0.639	0.360
			0.20	0.80	0.325	0.216	0.076
			0.40	1.00	0.491	0.354	0.145
Weibull($\beta = 1$)	Beta(2,3)	25	0.00	1.00	0.919	0.856	0.643
			0.00	0.80	0.655	0.532	0.280
			0.00	0.60	0.414	0.291	0.115
			0.20	1.00	0.573	0.436	0.195
			0.20	0.80	0.245	0.152	0.047
			0.40	1.00	0.334	0.217	0.074
Weibull($\beta = 1$)	Chi-square(1)	25	0.00	1.00	0.711	0.616	0.413
			0.00	0.80	0.631	0.526	0.326
			0.00	0.60	0.524	0.416	0.228
			0.20	1.00	0.341	0.243	0.103
			0.20	0.80	0.247	0.162	0.060
			0.40	1.00	0.181	0.111	0.035
Weibull($\beta = 1$)	Chi-square(4)	25	0.00	1.00	0.398	0.279	0.107
			0.00	0.80	0.319	0.211	0.071
			0.00	0.60	0.233	0.144	0.041
			0.20	1.00	0.151	0.086	0.022
			0.20	0.80	0.124	0.066	0.015
			0.40	1.00	0.109	0.058	0.013

Table C.9 Power of the Test: A_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Log-normal(0,1)	25	0.00	1.00	0.269	0.187	0.085
			0.00	0.80	0.120	0.060	0.013
			0.00	0.60	0.106	0.055	0.012
			0.20	1.00	0.299	0.216	0.103
			0.20	0.80	0.128	0.069	0.017
			0.40	1.00	0.263	0.185	0.083
Weibull($\beta = 1$)	Log-logistic(0,1)	25	0.00	1.00	0.920	0.891	0.812
			0.00	0.80	0.537	0.439	0.261
			0.00	0.60	0.243	0.160	0.058
			0.20	1.00	0.863	0.821	0.717
			0.20	0.80	0.380	0.281	0.137
			0.40	1.00	0.760	0.701	0.578
Weibull($\beta = 1$)	Log-double exp(0,1)	25	0.00	1.00	0.716	0.656	0.544
			0.00	0.80	0.199	0.122	0.042
			0.00	0.60	0.155	0.088	0.022
			0.20	1.00	0.747	0.689	0.574
			0.20	0.80	0.200	0.128	0.046
			0.40	1.00	0.735	0.672	0.547
Weibull($\beta = 1$)	Log-Cauchy(0,1)	25	0.00	1.00	0.979	0.974	0.959
			0.00	0.80	0.575	0.494	0.352
			0.00	0.60	0.333	0.247	0.132
			0.20	1.00	0.969	0.962	0.946
			0.20	0.80	0.412	0.334	0.217
			0.40	1.00	0.963	0.955	0.935

Appendix D. Power Study of A_s^2, H_0: Weibull($\beta = 2$)

Table D.1 Power of the Test: A_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Weibull($\beta = 1$)	5	0.00	1.00	0.197	0.115	0.032
			0.00	0.80	0.143	0.076	0.018
			0.00	0.60	0.110	0.054	0.010
			0.20	1.00	0.140	0.076	0.016
			0.20	0.80	0.106	0.054	0.011
			0.40	1.00	0.112	0.057	0.011
Weibull($\beta = 2$)	Weibull($\beta = 2$)	5	0.00	1.00	0.098	0.049	0.010
			0.00	0.80	0.100	0.050	0.010
			0.00	0.60	0.099	0.050	0.009
			0.20	1.00	0.101	0.051	0.010
			0.20	0.80	0.098	0.050	0.010
			0.40	1.00	0.102	0.050	0.010
Weibull($\beta = 2$)	Weibull($\beta = 3.5$)	5	0.00	1.00	0.115	0.059	0.012
			0.00	0.80	0.107	0.053	0.011
			0.00	0.60	0.102	0.050	0.010
			0.20	1.00	0.103	0.052	0.011
			0.20	0.80	0.097	0.048	0.011
			0.40	1.00	0.099	0.050	0.011
Weibull($\beta = 2$)	Gamma($\beta = 2$)	5	0.00	1.00	0.126	0.064	0.014
			0.00	0.80	0.109	0.055	0.011
			0.00	0.60	0.099	0.050	0.010
			0.20	1.00	0.114	0.058	0.012
			0.20	0.80	0.098	0.049	0.009
			0.40	1.00	0.107	0.054	0.011
Weibull($\beta = 2$)	Normal(0,1)	5	0.00	1.00	0.119	0.062	0.013
			0.00	0.80	0.108	0.054	0.011
			0.00	0.60	0.101	0.050	0.010
			0.20	1.00	0.102	0.053	0.011
			0.20	0.80	0.097	0.048	0.010
			0.40	1.00	0.101	0.050	0.010

Table D.2 Power of the Test: A_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Uniform(0,1)	5	0.00	1.00	0.132	0.069	0.015
			0.00	0.80	0.114	0.058	0.012
			0.00	0.60	0.108	0.053	0.010
			0.20	1.00	0.127	0.067	0.015
			0.20	0.80	0.105	0.053	0.011
			0.40	1.00	0.118	0.059	0.012
Weibull($\beta = 2$)	Beta(2,2)	5	0.00	1.00	0.110	0.055	0.011
			0.00	0.80	0.103	0.052	0.010
			0.00	0.60	0.101	0.050	0.010
			0.20	1.00	0.106	0.054	0.011
			0.20	0.80	0.098	0.048	0.010
			0.40	1.00	0.102	0.051	0.010
Weibull($\beta = 2$)	Beta(2,3)	5	0.00	1.00	0.097	0.047	0.009
			0.00	0.80	0.101	0.050	0.010
			0.00	0.60	0.098	0.049	0.009
			0.20	1.00	0.103	0.051	0.010
			0.20	0.80	0.099	0.049	0.010
			0.40	1.00	0.102	0.051	0.010
Weibull($\beta = 2$)	Chi-square(1)	5	0.00	1.00	0.376	0.273	0.124
			0.00	0.80	0.257	0.166	0.060
			0.00	0.60	0.170	0.096	0.025
			0.20	1.00	0.209	0.127	0.038
			0.20	0.80	0.134	0.072	0.017
			0.40	1.00	0.131	0.066	0.013
Weibull($\beta = 2$)	Chi-square(4)	5	0.00	1.00	0.129	0.066	0.015
			0.00	0.80	0.106	0.053	0.011
			0.00	0.60	0.101	0.050	0.009
			0.20	1.00	0.115	0.060	0.013
			0.20	0.80	0.099	0.050	0.010
			0.40	1.00	0.109	0.055	0.012

Table D.3 Power of the Test: A_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Log-normal(0,1)	5	0.00	1.00	0.272	0.179	0.066
			0.00	0.80	0.148	0.081	0.018
			0.00	0.60	0.108	0.054	0.010
			0.20	1.00	0.187	0.113	0.032
			0.20	0.80	0.110	0.056	0.011
			0.40	1.00	0.131	0.068	0.014
Weibull($\beta = 2$)	Log-logistic(0,1)	5	0.00	1.00	0.514	0.420	0.259
			0.00	0.80	0.258	0.166	0.059
			0.00	0.60	0.136	0.069	0.014
			0.20	1.00	0.355	0.266	0.132
			0.20	0.80	0.142	0.076	0.016
			0.40	1.00	0.216	0.131	0.038
Weibull($\beta = 2$)	Log-double exp(0,1)	5	0.00	1.00	0.393	0.308	0.180
			0.00	0.80	0.166	0.097	0.028
			0.00	0.60	0.109	0.053	0.010
			0.20	1.00	0.304	0.221	0.106
			0.20	0.80	0.118	0.061	0.013
			0.40	1.00	0.198	0.118	0.035
Weibull($\beta = 2$)	Log-Cauchy(0,1)	5	0.00	1.00	0.623	0.562	0.459
			0.00	0.80	0.301	0.223	0.123
			0.00	0.60	0.159	0.096	0.036
			0.20	1.00	0.524	0.460	0.361
			0.20	0.80	0.183	0.117	0.046
			0.40	1.00	0.414	0.340	0.239

Table D.4 Power of the Test: A_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Weibull($\beta = 1$)	15	0.00	1.00	0.600	0.494	0.292
			0.00	0.80	0.394	0.286	0.125
			0.00	0.60	0.257	0.167	0.055
			0.20	1.00	0.323	0.227	0.093
			0.20	0.80	0.174	0.101	0.030
			0.40	1.00	0.195	0.118	0.037
Weibull($\beta = 2$)	Weibull($\beta = 2$)	15	0.00	1.00	0.100	0.050	0.010
			0.00	0.80	0.100	0.051	0.011
			0.00	0.60	0.101	0.050	0.010
			0.20	1.00	0.100	0.051	0.010
			0.20	0.80	0.097	0.048	0.010
			0.40	1.00	0.097	0.048	0.009
Weibull($\beta = 2$)	Weibull($\beta = 3.5$)	15	0.00	1.00	0.251	0.161	0.054
			0.00	0.80	0.188	0.112	0.031
			0.00	0.60	0.148	0.082	0.019
			0.20	1.00	0.132	0.071	0.018
			0.20	0.80	0.113	0.057	0.012
			0.40	1.00	0.105	0.053	0.011
Weibull($\beta = 2$)	Gamma($\beta = 2$)	15	0.00	1.00	0.269	0.180	0.067
			0.00	0.80	0.158	0.091	0.024
			0.00	0.60	0.124	0.065	0.013
			0.20	1.00	0.179	0.107	0.032
			0.20	0.80	0.121	0.063	0.014
			0.40	1.00	0.144	0.080	0.020
Weibull($\beta = 2$)	Normal(0,1)	15	0.00	1.00	0.278	0.184	0.068
			0.00	0.80	0.219	0.139	0.044
			0.00	0.60	0.177	0.102	0.027
			0.20	1.00	0.123	0.068	0.017
			0.20	0.80	0.108	0.055	0.012
			0.40	1.00	0.100	0.051	0.010

Table D.5 Power of the Test: A_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Uniform(0,1)	15	0.00	1.00	0.268	0.169	0.053
			0.00	0.80	0.128	0.069	0.014
			0.00	0.60	0.138	0.074	0.018
			0.20	1.00	0.289	0.187	0.066
			0.20	0.80	0.118	0.061	0.013
			0.40	1.00	0.257	0.162	0.050
Weibull($\beta = 2$)	Beta(2,2)	15	0.00	1.00	0.208	0.123	0.033
			0.00	0.80	0.117	0.061	0.014
			0.00	0.60	0.106	0.054	0.010
			0.20	1.00	0.170	0.098	0.025
			0.20	0.80	0.109	0.055	0.012
			0.40	1.00	0.143	0.078	0.018
Weibull($\beta = 2$)	Beta(2,3)	15	0.00	1.00	0.109	0.055	0.011
			0.00	0.80	0.099	0.049	0.009
			0.00	0.60	0.100	0.051	0.010
			0.20	1.00	0.108	0.056	0.012
			0.20	0.80	0.101	0.050	0.010
			0.40	1.00	0.101	0.051	0.010
Weibull($\beta = 2$)	Chi-square(1)	15	0.00	1.00	0.926	0.887	0.769
			0.00	0.80	0.799	0.719	0.526
			0.00	0.60	0.625	0.520	0.312
			0.20	1.00	0.606	0.502	0.308
			0.20	0.80	0.354	0.250	0.110
			0.40	1.00	0.321	0.226	0.094
Weibull($\beta = 2$)	Chi-square(4)	15	0.00	1.00	0.266	0.180	0.067
			0.00	0.80	0.161	0.092	0.024
			0.00	0.60	0.122	0.064	0.015
			0.20	1.00	0.182	0.109	0.033
			0.20	0.80	0.119	0.063	0.014
			0.40	1.00	0.142	0.080	0.020

Table D.6 Power of the Test: A_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Log-normal(0,1)	15	0.00	1.00	0.762	0.689	0.522
			0.00	0.80	0.420	0.315	0.149
			0.00	0.60	0.221	0.138	0.043
			0.20	1.00	0.557	0.464	0.291
			0.20	0.80	0.231	0.147	0.050
			0.40	1.00	0.378	0.285	0.143
Weibull($\beta = 2$)	Log-logistic(0,1)	15	0.00	1.00	0.973	0.959	0.915
			0.00	0.80	0.779	0.702	0.519
			0.00	0.60	0.471	0.360	0.176
			0.20	1.00	0.881	0.841	0.741
			0.20	0.80	0.455	0.352	0.189
			0.40	1.00	0.724	0.651	0.508
Weibull($\beta = 2$)	Log-double exp(0,1)	15	0.00	1.00	0.878	0.842	0.758
			0.00	0.80	0.435	0.342	0.189
			0.00	0.60	0.177	0.106	0.031
			0.20	1.00	0.802	0.751	0.638
			0.20	0.80	0.292	0.205	0.093
			0.40	1.00	0.693	0.621	0.481
Weibull($\beta = 2$)	Log-Cauchy(0,1)	15	0.00	1.00	0.977	0.970	0.950
			0.00	0.80	0.724	0.652	0.504
			0.00	0.60	0.428	0.333	0.184
			0.20	1.00	0.943	0.927	0.894
			0.20	0.80	0.470	0.383	0.253
			0.40	1.00	0.908	0.886	0.839

Table D.7 Power of the Test: A_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Weibull($\beta = 1$)	25	0.00	1.00	0.838	0.765	0.577
			0.00	0.80	0.615	0.504	0.283
			0.00	0.60	0.425	0.310	0.138
			0.20	1.00	0.483	0.372	0.187
			0.20	0.80	0.241	0.155	0.053
			0.40	1.00	0.285	0.193	0.074
Weibull($\beta = 2$)	Weibull($\beta = 2$)	25	0.00	1.00	0.100	0.050	0.010
			0.00	0.80	0.097	0.048	0.009
			0.00	0.60	0.101	0.051	0.010
			0.20	1.00	0.098	0.048	0.009
			0.20	0.80	0.096	0.047	0.010
			0.40	1.00	0.098	0.048	0.010
Weibull($\beta = 2$)	Weibull($\beta = 3.5$)	25	0.00	1.00	0.391	0.278	0.116
			0.00	0.80	0.275	0.179	0.059
			0.00	0.60	0.209	0.128	0.037
			0.20	1.00	0.168	0.098	0.027
			0.20	0.80	0.125	0.067	0.015
			0.40	1.00	0.120	0.065	0.015
Weibull($\beta = 2$)	Gamma($\beta = 2$)	25	0.00	1.00	0.394	0.289	0.133
			0.00	0.80	0.215	0.136	0.044
			0.00	0.60	0.151	0.086	0.023
			0.20	1.00	0.250	0.163	0.057
			0.20	0.80	0.137	0.075	0.019
			0.40	1.00	0.180	0.108	0.031
Weibull($\beta = 2$)	Normal(0,1)	25	0.00	1.00	0.435	0.328	0.154
			0.00	0.80	0.340	0.240	0.099
			0.00	0.60	0.273	0.181	0.063
			0.20	1.00	0.150	0.084	0.021
			0.20	0.80	0.125	0.067	0.016
			0.40	1.00	0.110	0.058	0.013

Table D.8 Power of the Test: A_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Uniform(0,1)	25	0.00	1.00	0.439	0.301	0.113
			0.00	0.80	0.149	0.080	0.018
			0.00	0.60	0.176	0.101	0.027
			0.20	1.00	0.486	0.348	0.148
			0.20	0.80	0.126	0.068	0.016
			0.40	1.00	0.449	0.327	0.137
Weibull($\beta = 2$)	Beta(2,2)	25	0.00	1.00	0.326	0.213	0.070
			0.00	0.80	0.133	0.073	0.016
			0.00	0.60	0.108	0.055	0.011
			0.20	1.00	0.258	0.160	0.049
			0.20	0.80	0.119	0.062	0.013
			0.40	1.00	0.208	0.124	0.035
Weibull($\beta = 2$)	Beta(2,3)	25	0.00	1.00	0.126	0.066	0.015
			0.00	0.80	0.100	0.050	0.010
			0.00	0.60	0.099	0.050	0.010
			0.20	1.00	0.127	0.068	0.014
			0.20	0.80	0.101	0.049	0.010
			0.40	1.00	0.122	0.065	0.014
Weibull($\beta = 2$)	Chi-square(1)	25	0.00	1.00	0.995	0.991	0.970
			0.00	0.80	0.966	0.942	0.852
			0.00	0.60	0.881	0.820	0.650
			0.20	1.00	0.825	0.751	0.568
			0.20	0.80	0.536	0.426	0.233
			0.40	1.00	0.491	0.386	0.203
Weibull($\beta = 2$)	Chi-square(4)	25	0.00	1.00	0.396	0.292	0.137
			0.00	0.80	0.217	0.138	0.044
			0.00	0.60	0.154	0.088	0.022
			0.20	1.00	0.252	0.164	0.057
			0.20	0.80	0.139	0.078	0.020
			0.40	1.00	0.180	0.107	0.030

Table D.9 Power of the Test: A_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Log-normal(0,1)	25	0.00	1.00	0.937	0.904	0.808
			0.00	0.80	0.640	0.536	0.328
			0.00	0.60	0.346	0.243	0.099
			0.20	1.00	0.779	0.702	0.531
			0.20	0.80	0.346	0.243	0.103
			0.40	1.00	0.582	0.485	0.309
Weibull($\beta = 2$)	Log-logistic(0,1)	25	0.00	1.00	0.999	0.998	0.994
			0.00	0.80	0.952	0.922	0.828
			0.00	0.60	0.725	0.628	0.410
			0.20	1.00	0.982	0.971	0.938
			0.20	0.80	0.680	0.584	0.388
			0.40	1.00	0.914	0.880	0.794
Weibull($\beta = 2$	Log-double exp(0,1)	25	0.00	1.00	0.976	0.966	0.938
			0.00	0.80	0.616	0.527	0.351
			0.00	0.60	0.249	0.162	0.056
			0.20	1.00	0.945	0.923	0.867
			0.20	0.80	0.421	0.327	0.177
			0.40	1.00	0.896	0.860	0.769
Weibull($\beta = 2$)	Log-Cauchy(0,1)	25	0.00	1.00	0.999	0.998	0.996
			0.00	0.80	0.912	0.874	0.771
			0.00	0.60	0.645	0.549	0.359
			0.20	1.00	0.994	0.991	0.984
			0.20	0.80	0.638	0.561	0.413
			0.40	1.00	0.986	0.981	0.970

Appendix E. Power Study of A_s^2, H_0: Weibull($\beta = 3.5$)

Table E.1 Power of the Test: A_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Weibull($\beta = 1$)	5	0.00	1.00	0.272	0.173	0.058
			0.00	0.80	0.173	0.098	0.025
			0.00	0.60	0.124	0.063	0.012
			0.20	1.00	0.168	0.094	0.023
			0.20	0.80	0.116	0.058	0.012
			0.40	1.00	0.118	0.060	0.012
Weibull($\beta = 3.5$)	Weibull($\beta = 2$)	5	0.00	1.00	0.116	0.059	0.012
			0.00	0.80	0.102	0.051	0.011
			0.00	0.60	0.100	0.050	0.010
			0.20	1.00	0.109	0.056	0.012
			0.20	0.80	0.102	0.050	0.010
			0.40	1.00	0.103	0.053	0.011
Weibull($\beta = 3.5$)	Weibull($\beta = 3.5$)	5	0.00	1.00	0.098	0.049	0.010
			0.00	0.80	0.102	0.051	0.010
			0.00	0.60	0.101	0.051	0.010
			0.20	1.00	0.105	0.053	0.010
			0.20	0.80	0.100	0.049	0.011
			0.40	1.00	0.100	0.051	0.011
Weibull($\beta = 3.5$)	Gamma($\beta = 2$)	5	0.00	1.00	0.179	0.101	0.027
			0.00	0.80	0.123	0.065	0.013
			0.00	0.60	0.106	0.053	0.010
			0.20	1.00	0.130	0.069	0.015
			0.20	0.80	0.107	0.054	0.011
			0.40	1.00	0.110	0.057	0.012
Weibull($\beta = 3.5$)	Normal(0,1)	5	0.00	1.00	0.104	0.052	0.011
			0.00	0.80	0.099	0.050	0.010
			0.00	0.60	0.103	0.051	0.010
			0.20	1.00	0.103	0.052	0.011
			0.20	0.80	0.100	0.049	0.010
			0.40	1.00	0.101	0.050	0.010

Table E.2 Power of the Test: A_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Uniform(0,1)	5	0.00	1.00	0.123	0.062	0.013
			0.00	0.80	0.121	0.061	0.013
			0.00	0.60	0.115	0.059	0.011
			0.20	1.00	0.120	0.062	0.013
			0.20	0.80	0.106	0.053	0.011
			0.40	1.00	0.111	0.057	0.012
Weibull($\beta = 3.5$)	Beta(2,2)	5	0.00	1.00	0.101	0.049	0.011
			0.00	0.80	0.102	0.051	0.011
			0.00	0.60	0.103	0.052	0.010
			0.20	1.00	0.103	0.051	0.010
			0.20	0.80	0.103	0.051	0.010
			0.40	1.00	0.100	0.050	0.010
Weibull($\beta = 3.5$)	Beta(2,3)	5	0.00	1.00	0.103	0.052	0.011
			0.00	0.80	0.104	0.052	0.011
			0.00	0.60	0.105	0.052	0.010
			0.20	1.00	0.100	0.050	0.011
			0.20	0.80	0.101	0.051	0.010
			0.40	1.00	0.101	0.051	0.011
Weibull($\beta = 3.5$)	Chi-square(1)	5	0.00	1.00	0.465	0.352	0.179
			0.00	0.80	0.301	0.205	0.079
			0.00	0.60	0.190	0.110	0.029
			0.20	1.00	0.247	0.156	0.050
			0.20	0.80	0.146	0.077	0.018
			0.40	1.00	0.140	0.073	0.016
Weibull($\beta = 3.5$)	Chi-square(4)	5	0.00	1.00	0.179	0.103	0.027
			0.00	0.80	0.123	0.064	0.014
			0.00	0.60	0.105	0.052	0.009
			0.20	1.00	0.133	0.071	0.015
			0.20	0.80	0.107	0.053	0.011
			0.40	1.00	0.109	0.057	0.011

Table E.3 Power of the Test: A_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Log-normal(0,1)	5	0.00	1.00	0.356	0.253	0.109
			0.00	0.80	0.181	0.105	0.025
			0.00	0.60	0.119	0.060	0.011
			0.20	1.00	0.222	0.139	0.043
			0.20	0.80	0.119	0.060	0.012
			0.40	1.00	0.139	0.073	0.015
Weibull($\beta = 3.5$)	Log-logistic(0,1)	5	0.00	1.00	0.593	0.498	0.328
			0.00	0.80	0.308	0.208	0.078
			0.00	0.60	0.153	0.079	0.017
			0.20	1.00	0.399	0.304	0.158
			0.20	0.80	0.159	0.084	0.018
			0.40	1.00	0.226	0.141	0.043
Weibull($\beta = 3.5$)	Log-double exp(0,1)	5	0.00	1.00	0.470	0.380	0.236
			0.00	0.80	0.202	0.123	0.038
			0.00	0.60	0.117	0.059	0.012
			0.20	1.00	0.344	0.254	0.125
			0.20	0.80	0.129	0.066	0.014
			0.40	1.00	0.214	0.130	0.038
Weibull($\beta = 3.5$)	Log-Cauchy(0,1)	5	0.00	1.00	0.673	0.610	0.504
			0.00	0.80	0.336	0.252	0.139
			0.00	0.60	0.174	0.108	0.039
			0.20	1.00	0.556	0.490	0.383
			0.20	0.80	0.194	0.122	0.047
			0.40	1.00	0.424	0.352	0.248

Table E.4 Power of the Test: A_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Weibull($\beta = 1$)	15	0.00	1.00	0.832	0.755	0.558
			0.00	0.80	0.594	0.479	0.264
			0.00	0.60	0.382	0.270	0.111
			0.20	1.00	0.493	0.387	0.197
			0.20	0.80	0.240	0.155	0.050
			0.40	1.00	0.281	0.188	0.069
Weibull($\beta = 3.5$)	Weibull($\beta = 2$)	15	0.00	1.00	0.223	0.139	0.043
			0.00	0.80	0.154	0.088	0.022
			0.00	0.60	0.124	0.066	0.014
			0.20	1.00	0.152	0.085	0.020
			0.20	0.80	0.112	0.058	0.013
			0.40	1.00	0.121	0.063	0.014
Weibull($\beta = 3.5$)	Weibull($\beta = 3.5$)	15	0.00	1.00	0.100	0.050	0.010
			0.00	0.80	0.102	0.051	0.010
			0.00	0.60	0.102	0.051	0.011
			0.20	1.00	0.102	0.052	0.010
			0.20	0.80	0.100	0.050	0.010
			0.40	1.00	0.099	0.051	0.010
Weibull($\beta = 3.5$)	Gamma($\beta = 2$)	15	0.00	1.00	0.562	0.451	0.242
			0.00	0.80	0.322	0.220	0.083
			0.00	0.60	0.198	0.117	0.033
			0.20	1.00	0.325	0.226	0.086
			0.20	0.80	0.161	0.093	0.026
			0.40	1.00	0.205	0.127	0.037
Weibull($\beta = 3.5$)	Normal(0,1)	15	0.00	1.00	0.116	0.059	0.013
			0.00	0.80	0.111	0.057	0.012
			0.00	0.60	0.109	0.056	0.012
			0.20	1.00	0.108	0.055	0.012
			0.20	0.80	0.099	0.049	0.010
			0.40	1.00	0.105	0.053	0.010

Table E.5 Power of the Test: A_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Uniform(0,1)	15	0.00	1.00	0.183	0.099	0.022
			0.00	0.80	0.192	0.111	0.032
			0.00	0.60	0.197	0.117	0.036
			0.20	1.00	0.190	0.110	0.030
			0.20	0.80	0.107	0.055	0.011
			0.40	1.00	0.204	0.122	0.034
Weibull($\beta = 3.5$)	Beta(2,2)	15	0.00	1.00	0.092	0.044	0.008
			0.00	0.80	0.109	0.054	0.012
			0.00	0.60	0.109	0.056	0.013
			0.20	1.00	0.103	0.052	0.010
			0.20	0.80	0.098	0.048	0.010
			0.40	1.00	0.107	0.055	0.011
Weibull($\beta = 3.5$)	Beta(2,3)	15	0.00	1.00	0.126	0.065	0.013
			0.00	0.80	0.142	0.077	0.018
			0.00	0.60	0.131	0.069	0.016
			0.20	1.00	0.094	0.047	0.009
			0.20	0.80	0.104	0.052	0.012
			0.40	1.00	0.095	0.047	0.008
Weibull($\beta = 3.5$)	Chi-square(1)	15	0.00	1.00	0.979	0.963	0.899
			0.00	0.80	0.895	0.838	0.680
			0.00	0.60	0.735	0.637	0.427
			0.20	1.00	0.747	0.662	0.460
			0.20	0.80	0.442	0.332	0.160
			0.40	1.00	0.429	0.325	0.154
Weibull($\beta = 3.5$)	Chi-square(4)	15	0.00	1.00	0.563	0.450	0.243
			0.00	0.80	0.320	0.218	0.082
			0.00	0.60	0.198	0.117	0.034
			0.20	1.00	0.323	0.225	0.088
			0.20	0.80	0.164	0.095	0.026
			0.40	1.00	0.204	0.126	0.037

Table E.6 Power of the Test: A_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Log-normal(0,1)	15	0.00	1.00	0.910	0.866	0.741
			0.00	0.80	0.623	0.514	0.300
			0.00	0.60	0.343	0.239	0.094
			0.20	1.00	0.706	0.621	0.435
			0.20	0.80	0.310	0.214	0.083
			0.40	1.00	0.476	0.379	0.208
Weibull($\beta = 3.5$)	Log-logistic(0,1)	15	0.00	1.00	0.992	0.987	0.965
			0.00	0.80	0.888	0.831	0.681
			0.00	0.60	0.602	0.491	0.279
			0.20	1.00	0.932	0.904	0.823
			0.20	0.80	0.547	0.443	0.259
			0.40	1.00	0.788	0.726	0.587
Weibull($\beta = 3.5$)	Log-double exp(0,1)	15	0.00	1.00	0.951	0.929	0.872
			0.00	0.80	0.606	0.511	0.329
			0.00	0.60	0.270	0.177	0.065
			0.20	1.00	0.873	0.833	0.731
			0.20	0.80	0.368	0.274	0.134
			0.40	1.00	0.758	0.694	0.551
Weibull($\beta = 3.5$)	Log-cauchy(0,1)	15	0.00	1.00	0.992	0.988	0.975
			0.00	0.80	0.839	0.779	0.636
			0.00	0.60	0.532	0.427	0.249
			0.20	1.00	0.965	0.953	0.923
			0.20	0.80	0.538	0.456	0.307
			0.40	1.00	0.930	0.911	0.865

Table E.7 Power of the Test: A_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Weibull($\beta = 1$)	25	0.00	1.00	0.976	0.957	0.887
			0.00	0.80	0.853	0.775	0.580
			0.00	0.60	0.643	0.524	0.294
			0.20	1.00	0.722	0.627	0.415
			0.20	0.80	0.369	0.264	0.112
			0.40	1.00	0.432	0.325	0.156
Weibull($\beta = 3.5$)	Weibull($\beta = 2$)	25	0.00	1.00	0.355	0.250	0.099
			0.00	0.80	0.225	0.138	0.043
			0.00	0.60	0.160	0.091	0.024
			0.20	1.00	0.188	0.110	0.033
			0.20	0.80	0.126	0.069	0.016
			0.40	1.00	0.137	0.075	0.019
Weibull($\beta = 3.5$)	Weibull($\beta = 3.5$)	25	0.00	1.00	0.103	0.053	0.011
			0.00	0.80	0.099	0.050	0.011
			0.00	0.60	0.103	0.051	0.011
			0.20	1.00	0.099	0.050	0.010
			0.20	0.80	0.100	0.053	0.010
			0.40	1.00	0.098	0.049	0.010
Weibull($\beta = 3.5$)	Gamma($\beta = 2$)	25	0.00	1.00	0.814	0.733	0.534
			0.00	0.80	0.518	0.399	0.202
			0.00	0.60	0.312	0.209	0.075
			0.20	1.00	0.494	0.384	0.194
			0.20	0.80	0.223	0.142	0.045
			0.40	1.00	0.303	0.209	0.080
Weibull($\beta = 3.5$)	Normal(0,1)	25	0.00	1.00	0.128	0.067	0.015
			0.00	0.80	0.121	0.062	0.015
			0.00	0.60	0.117	0.062	0.014
			0.20	1.00	0.108	0.055	0.011
			0.20	0.80	0.103	0.052	0.011
			0.40	1.00	0.109	0.057	0.012

Table E.8 Power of the Test: A_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Uniform(0,1)	25	0.00	1.00	0.327	0.201	0.056
			0.00	0.80	0.303	0.194	0.067
			0.00	0.60	0.324	0.216	0.078
			0.20	1.00	0.290	0.182	0.058
			0.20	0.80	0.108	0.056	0.012
			0.40	1.00	0.326	0.219	0.078
Weibull($\beta = 3.5$)	Beta(2,2)	25	0.00	1.00	0.103	0.050	0.009
			0.00	0.80	0.126	0.065	0.015
			0.00	0.60	0.132	0.070	0.017
			0.20	1.00	0.118	0.060	0.012
			0.20	0.80	0.098	0.049	0.010
			0.40	1.00	0.129	0.069	0.015
Weibull($\beta = 3.5$)	Beta(2,3)	25	0.00	1.00	0.177	0.102	0.025
			0.00	0.80	0.200	0.119	0.035
			0.00	0.60	0.166	0.095	0.026
			0.20	1.00	0.096	0.046	0.008
			0.20	0.80	0.111	0.059	0.013
			0.40	1.00	0.091	0.044	0.008
Weibull($\beta = 3.5$)	Chi-square(1)	25	0.00	1.00	1.000	0.999	0.997
			0.00	0.80	0.993	0.985	0.952
			0.00	0.60	0.948	0.910	0.789
			0.20	1.00	0.932	0.893	0.770
			0.20	0.80	0.668	0.565	0.352
			0.40	1.00	0.641	0.541	0.338
Weibull($\beta = 3.5$)	Chi-square(4)	25	0.00	1.00	0.812	0.732	0.532
			0.00	0.80	0.516	0.397	0.201
			0.00	0.60	0.312	0.210	0.073
			0.20	1.00	0.493	0.382	0.192
			0.20	0.80	0.226	0.142	0.047
			0.40	1.00	0.297	0.206	0.081

Table E.9 Power of the Test: A_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Log-normal(0,1)	25	0.00	1.00	0.993	0.987	0.962
			0.00	0.80	0.869	0.801	0.626
			0.00	0.60	0.573	0.454	0.237
			0.20	1.00	0.905	0.861	0.735
			0.20	0.80	0.487	0.377	0.190
			0.40	1.00	0.711	0.627	0.446
Weibull($\beta = 3.5$)	Log-logistic(0,1)	25	0.00	1.00	1.000	1.000	1.000
			0.00	0.80	0.989	0.980	0.943
			0.00	0.60	0.866	0.794	0.605
			0.20	1.00	0.994	0.990	0.974
			0.20	0.80	0.787	0.705	0.516
			0.40	1.00	0.949	0.926	0.862
Weibull($\beta = 3.5$)	Log-double exp(0,1)	25	0.00	1.00	0.997	0.995	0.987
			0.00	0.80	0.832	0.762	0.606
			0.00	0.60	0.427	0.314	0.145
			0.20	1.00	0.978	0.968	0.937
			0.20	0.80	0.544	0.450	0.274
			0.40	1.00	0.939	0.913	0.842
Weibull($\beta = 3.5$)	Log-Cauchy(0,1)	25	0.00	1.00	1.000	1.000	0.999
			0.00	0.80	0.971	0.952	0.895
			0.00	0.60	0.788	0.705	0.514
			0.20	1.00	0.997	0.996	0.992
			0.20	0.80	0.733	0.663	0.515
			0.40	1.00	0.992	0.989	0.980

Appendix F. Power Study of W_s^2, H_0: Weibull($\beta = 1$)

Table F.1 Power of the Test: W_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Weibull($\beta = 1$)	5	0.00	1.00	0.102	0.051	0.009
			0.00	0.80	0.101	0.050	0.010
			0.00	0.60	0.099	0.051	0.010
			0.20	1.00	0.100	0.051	0.010
			0.20	0.80	0.101	0.050	0.010
			0.40	1.00	0.098	0.049	0.011
Weibull($\beta = 1$)	Weibull($\beta = 2$)	5	0.00	1.00	0.180	0.099	0.022
			0.00	0.80	0.130	0.067	0.015
			0.00	0.60	0.105	0.053	0.011
			0.20	1.00	0.101	0.051	0.010
			0.20	0.80	0.098	0.050	0.010
			0.40	1.00	0.097	0.047	0.010
Weibull($\beta = 1$)	Weibull($\beta = 3.5$)	5	0.00	1.00	0.267	0.163	0.041
			0.00	0.80	0.171	0.093	0.021
			0.00	0.60	0.116	0.059	0.012
			0.20	1.00	0.121	0.063	0.012
			0.20	0.80	0.101	0.051	0.010
			0.40	1.00	0.096	0.047	0.010
Weibull($\beta = 1$)	Gamma($\beta = 2$)	5	0.00	1.00	0.121	0.063	0.012
			0.00	0.80	0.109	0.055	0.010
			0.00	0.60	0.098	0.048	0.009
			0.20	1.00	0.094	0.046	0.009
			0.20	0.80	0.098	0.049	0.009
			0.40	1.00	0.097	0.048	0.010
Weibull($\beta = 1$)	Normal(0,1)	5	0.00	1.00	0.270	0.167	0.043
			0.00	0.80	0.177	0.098	0.022
			0.00	0.60	0.116	0.059	0.013
			0.20	1.00	0.119	0.062	0.012
			0.20	0.80	0.099	0.049	0.011
			0.40	1.00	0.096	0.048	0.009

Table F.2 Power of the Test: W_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Uniform(0,1)	5	0.00	1.00	0.220	0.137	0.039
			0.00	0.80	0.139	0.076	0.017
			0.00	0.60	0.109	0.055	0.010
			0.20	1.00	0.152	0.087	0.020
			0.20	0.80	0.106	0.054	0.012
			0.40	1.00	0.119	0.061	0.013
Weibull($\beta = 1$)	Beta(2,2)	5	0.00	1.00	0.242	0.148	0.036
			0.00	0.80	0.156	0.085	0.019
			0.00	0.60	0.110	0.054	0.011
			0.20	1.00	0.129	0.068	0.014
			0.20	0.80	0.104	0.053	0.011
			0.40	1.00	0.103	0.051	0.010
Weibull($\beta = 1$)	Beta(2,3)	5	0.00	1.00	0.198	0.112	0.025
			0.00	0.80	0.136	0.071	0.016
			0.00	0.60	0.106	0.053	0.010
			0.20	1.00	0.112	0.058	0.012
			0.20	0.80	0.099	0.050	0.009
			0.40	1.00	0.099	0.049	0.010
Weibull($\beta = 1$)	Chi-square(1)	5	0.00	1.00	0.151	0.090	0.028
			0.00	0.80	0.144	0.085	0.025
			0.00	0.60	0.131	0.071	0.017
			0.20	1.00	0.132	0.075	0.020
			0.20	0.80	0.117	0.061	0.012
			0.40	1.00	0.111	0.057	0.013
Weibull($\beta = 1$)	Chi-square(4)	5	0.00	1.00	0.122	0.064	0.013
			0.00	0.80	0.111	0.057	0.012
			0.00	0.60	0.100	0.050	0.010
			0.20	1.00	0.094	0.048	0.009
			0.20	0.80	0.096	0.048	0.010
			0.40	1.00	0.098	0.049	0.010

Table F.3 Power of the Test: W_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Log-normal(0,1)	5	0.00	1.00	0.129	0.070	0.016
			0.00	0.80	0.100	0.049	0.010
			0.00	0.60	0.095	0.048	0.010
			0.20	1.00	0.127	0.069	0.017
			0.20	0.80	0.101	0.051	0.010
			0.40	1.00	0.110	0.055	0.011
Weibull($\beta = 1$)	Log-logistic(0,1)	5	0.00	1.00	0.295	0.220	0.116
			0.00	0.80	0.149	0.086	0.024
			0.00	0.60	0.105	0.053	0.010
			0.20	1.00	0.257	0.183	0.083
			0.20	0.80	0.125	0.065	0.014
			0.40	1.00	0.177	0.104	0.030
Weibull($\beta = 1$)	Log-double exp(0,1)	5	0.00	1.00	0.245	0.173	0.082
			0.00	0.80	0.124	0.067	0.016
			0.00	0.60	0.101	0.052	0.011
			0.20	1.00	0.225	0.155	0.067
			0.20	0.80	0.108	0.054	0.012
			0.40	1.00	0.169	0.097	0.026
Weibull($\beta = 1$)	Log-Cauchy(0,1)	5	0.00	1.00	0.484	0.429	0.347
			0.00	0.80	0.201	0.139	0.071
			0.00	0.60	0.139	0.083	0.031
			0.20	1.00	0.455	0.399	0.313
			0.20	0.80	0.162	0.101	0.039
			0.40	1.00	0.380	0.314	0.224

Table F.4 Power of the Test: W_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Weibull($\beta = 1$)	15	0.00	1.00	0.102	0.051	0.010
			0.00	0.80	0.100	0.050	0.010
			0.00	0.60	0.101	0.051	0.011
			0.20	1.00	0.101	0.051	0.009
			0.20	0.80	0.098	0.049	0.010
			0.40	1.00	0.098	0.050	0.011
Weibull($\beta = 1$)	Weibull($\beta = 2$)	15	0.00	1.00	0.566	0.430	0.199
			0.00	0.80	0.379	0.262	0.094
			0.00	0.60	0.250	0.155	0.050
			0.20	1.00	0.242	0.149	0.044
			0.20	0.80	0.147	0.082	0.021
			0.40	1.00	0.140	0.077	0.018
Weibull($\beta = 1$)	Weibull($\beta = 3.5$)	15	0.00	1.00	0.836	0.746	0.519
			0.00	0.80	0.634	0.514	0.273
			0.00	0.60	0.425	0.307	0.131
			0.20	1.00	0.398	0.278	0.105
			0.20	0.80	0.213	0.130	0.039
			0.40	1.00	0.194	0.116	0.031
Weibull($\beta = 1$)	Gamma($\beta = 2$)	15	0.00	1.00	0.239	0.149	0.043
			0.00	0.80	0.190	0.113	0.030
			0.00	0.60	0.155	0.087	0.023
			0.20	1.00	0.128	0.068	0.016
			0.20	0.80	0.107	0.055	0.011
			0.40	1.00	0.097	0.049	0.010
Weibull($\beta = 1$)	Normal(0,1)	15	0.00	1.00	0.841	0.760	0.548
			0.00	0.80	0.670	0.556	0.326
			0.00	0.60	0.473	0.355	0.168
			0.20	1.00	0.381	0.262	0.096
			0.20	0.80	0.207	0.126	0.038
			0.40	1.00	0.178	0.104	0.028

Table F.5 Power of the Test: W_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Uniform(0,1)	15	0.00	1.00	0.630	0.493	0.242
			0.00	0.80	0.281	0.182	0.060
			0.00	0.60	0.149	0.085	0.022
			0.20	1.00	0.516	0.382	0.161
			0.20	0.80	0.195	0.116	0.034
			0.40	1.00	0.372	0.254	0.091
Weibull($\beta = 1$)	Beta(2,2)	15	0.00	1.00	0.773	0.654	0.381
			0.00	0.80	0.482	0.355	0.146
			0.00	0.60	0.281	0.181	0.061
			0.20	1.00	0.446	0.315	0.121
			0.20	0.80	0.199	0.118	0.035
			0.40	1.00	0.261	0.163	0.049
Weibull($\beta = 1$)	Beta(2,3)	15	0.00	1.00	0.634	0.498	0.244
			0.00	0.80	0.388	0.267	0.096
			0.00	0.60	0.243	0.152	0.046
			0.20	1.00	0.327	0.214	0.071
			0.20	0.80	0.164	0.092	0.025
			0.40	1.00	0.187	0.109	0.030
Weibull($\beta = 1$)	Chi-square(1)	15	0.00	1.00	0.427	0.320	0.155
			0.00	0.80	0.360	0.257	0.110
			0.00	0.60	0.290	0.200	0.084
			0.20	1.00	0.229	0.150	0.053
			0.20	0.80	0.175	0.106	0.034
			0.40	1.00	0.146	0.084	0.022
Weibull($\beta = 1$)	Chi-square(4)	15	0.00	1.00	0.243	0.152	0.047
			0.00	0.80	0.192	0.113	0.029
			0.00	0.60	0.155	0.087	0.023
			0.20	1.00	0.125	0.068	0.016
			0.20	0.80	0.106	0.055	0.012
			0.40	1.00	0.099	0.051	0.011

Table F.6 Power of the Test: W_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Log-normal(0,1)	15	0.00	1.00	0.228	0.154	0.066
			0.00	0.80	0.116	0.061	0.014
			0.00	0.60	0.103	0.054	0.011
			0.20	1.00	0.232	0.158	0.065
			0.20	0.80	0.116	0.061	0.014
			0.40	1.00	0.197	0.129	0.049
Weibull($\beta = 1$)	Log-logistic(0,1)	15	0.00	1.00	0.764	0.704	0.579
			0.00	0.80	0.375	0.281	0.143
			0.00	0.60	0.184	0.113	0.037
			0.20	1.00	0.676	0.604	0.469
			0.20	0.80	0.262	0.179	0.075
			0.40	1.00	0.555	0.477	0.339
Weibull($\beta = 1$)	Log-double exp(0,1)	15	0.00	1.00	0.576	0.505	0.389
			0.00	0.80	0.193	0.121	0.040
			0.00	0.60	0.130	0.069	0.017
			0.20	1.00	0.585	0.514	0.388
			0.20	0.80	0.179	0.111	0.037
			0.40	1.00	0.531	0.454	0.320
Weibull($\beta = 1$)	Log-Cauchy(0,1)	15	0.00	1.00	0.891	0.870	0.831
			0.00	0.80	0.376	0.301	0.196
			0.00	0.60	0.161	0.095	0.032
			0.20	1.00	0.877	0.854	0.811
			0.20	0.80	0.331	0.257	0.159
			0.40	1.00	0.858	0.830	0.775

Table F.7 Power of the Test: W_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Weibull($\beta = 1$)	25	0.00	1.00	0.101	0.052	0.010
			0.00	0.80	0.102	0.051	0.011
			0.00	0.60	0.100	0.049	0.010
			0.20	1.00	0.102	0.050	0.010
			0.20	0.80	0.100	0.049	0.010
			0.40	1.00	0.099	0.049	0.009
Weibull($\beta = 1$)	Weibull($\beta = 2$)	25	0.00	1.00	0.843	0.751	0.515
			0.00	0.80	0.652	0.531	0.283
			0.00	0.60	0.449	0.326	0.136
			0.20	1.00	0.398	0.277	0.103
			0.20	0.80	0.217	0.132	0.039
			0.40	1.00	0.203	0.121	0.032
Weibull($\beta = 1$)	Weibull($\beta = 3.5$)	25	0.00	1.00	0.982	0.962	0.882
			0.00	0.80	0.900	0.840	0.664
			0.00	0.60	0.725	0.616	0.379
			0.20	1.00	0.644	0.514	0.265
			0.20	0.80	0.347	0.237	0.086
			0.40	1.00	0.319	0.210	0.068
Weibull($\beta = 1$)	Gamma($\beta = 2$)	25	0.00	1.00	0.412	0.289	0.110
			0.00	0.80	0.328	0.218	0.075
			0.00	0.60	0.246	0.154	0.048
			0.20	1.00	0.158	0.088	0.023
			0.20	0.80	0.128	0.067	0.016
			0.40	1.00	0.113	0.059	0.013
Weibull($\beta = 1$)	Normal(0,1)	25	0.00	1.00	0.983	0.967	0.897
			0.00	0.80	0.917	0.868	0.714
			0.00	0.60	0.770	0.678	0.462
			0.20	1.00	0.610	0.480	0.242
			0.20	0.80	0.341	0.232	0.087
			0.40	1.00	0.285	0.183	0.058

Table F.8 Power of the Test: W_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Uniform(0,1)	25	0.00	1.00	0.884	0.796	0.544
			0.00	0.80	0.457	0.332	0.141
			0.00	0.60	0.204	0.125	0.036
			0.20	1.00	0.784	0.664	0.397
			0.20	0.80	0.306	0.200	0.070
			0.40	1.00	0.635	0.497	0.239
Weibull($\beta = 1$)	Beta(2,2)	25	0.00	1.00	0.968	0.933	0.788
			0.00	0.80	0.775	0.665	0.409
			0.00	0.60	0.501	0.373	0.167
			0.20	1.00	0.708	0.581	0.314
			0.20	0.80	0.322	0.214	0.075
			0.40	1.00	0.455	0.322	0.122
Weibull($\beta = 1$)	Beta(2,3)	25	0.00	1.00	0.903	0.831	0.605
			0.00	0.80	0.659	0.533	0.283
			0.00	0.60	0.427	0.304	0.121
			0.20	1.00	0.543	0.408	0.180
			0.20	0.80	0.245	0.153	0.046
			0.40	1.00	0.319	0.207	0.066
Weibull($\beta = 1$)	Chi-square(1)	25	0.00	1.00	0.640	0.532	0.318
			0.00	0.80	0.558	0.445	0.244
			0.00	0.60	0.458	0.346	0.166
			0.20	1.00	0.320	0.223	0.092
			0.20	0.80	0.234	0.150	0.053
			0.40	1.00	0.179	0.109	0.033
Weibull($\beta = 1$)	Chi-square(4)	25	0.00	1.00	0.410	0.287	0.112
			0.00	0.80	0.331	0.220	0.077
			0.00	0.60	0.246	0.153	0.046
			0.20	1.00	0.154	0.087	0.024
			0.20	0.80	0.127	0.068	0.015
			0.40	1.00	0.111	0.058	0.013

Table F.9 Power of the Test: W_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 1$)	Log-normal(0,1)	25	0.00	1.00	0.288	0.204	0.097
			0.00	0.80	0.132	0.069	0.016
			0.00	0.60	0.117	0.060	0.013
			0.20	1.00	0.308	0.225	0.114
			0.20	0.80	0.133	0.072	0.018
			0.40	1.00	0.271	0.193	0.089
Weibull($\beta = 1$)	Log-logistic(0,1)	25	0.00	1.00	0.920	0.891	0.812
			0.00	0.80	0.540	0.442	0.264
			0.00	0.60	0.247	0.163	0.060
			0.20	1.00	0.863	0.821	0.718
			0.20	0.80	0.385	0.286	0.138
			0.40	1.00	0.761	0.700	0.574
Weibull($\beta = 1$)	Log-double exp(0,1)	25	0.00	1.00	0.732	0.673	0.569
			0.00	0.80	0.218	0.137	0.051
			0.00	0.60	0.163	0.093	0.024
			0.20	1.00	0.761	0.705	0.597
			0.20	0.80	0.218	0.144	0.053
			0.40	1.00	0.742	0.679	0.552
Weibull($\beta = 1$)	Log-Cauchy(0,1)	25	0.00	1.00	0.976	0.970	0.956
			0.00	0.80	0.494	0.415	0.291
			0.00	0.60	0.185	0.114	0.038
			0.20	1.00	0.971	0.964	0.949
			0.20	0.80	0.429	0.351	0.232
			0.40	1.00	0.965	0.956	0.935

Appendix G. Power Study of W_s^2, H_0: *Weibull*($\beta = 2$)

Table G.1 Power of the Test: W_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Weibull($\beta = 1$)	5	0.00	1.00	0.191	0.113	0.033
			0.00	0.80	0.141	0.078	0.018
			0.00	0.60	0.110	0.054	0.010
			0.20	1.00	0.141	0.076	0.017
			0.20	0.80	0.106	0.054	0.011
			0.40	1.00	0.112	0.057	0.011
Weibull($\beta = 2$)	Weibull($\beta = 2$)	5	0.00	1.00	0.099	0.048	0.010
			0.00	0.80	0.099	0.050	0.010
			0.00	0.60	0.099	0.050	0.009
			0.20	1.00	0.102	0.050	0.010
			0.20	0.80	0.098	0.050	0.010
			0.40	1.00	0.102	0.050	0.010
Weibull($\beta = 2$)	Weibull($\beta = 3.5$)	5	0.00	1.00	0.119	0.060	0.013
			0.00	0.80	0.106	0.054	0.011
			0.00	0.60	0.102	0.050	0.010
			0.20	1.00	0.103	0.052	0.012
			0.20	0.80	0.097	0.048	0.011
			0.40	1.00	0.099	0.050	0.011
Weibull($\beta = 2$)	Gamma($\beta = 2$)	5	0.00	1.00	0.130	0.068	0.015
			0.00	0.80	0.109	0.056	0.011
			0.00	0.60	0.099	0.050	0.010
			0.20	1.00	0.115	0.059	0.013
			0.20	0.80	0.098	0.049	0.009
			0.40	1.00	0.107	0.054	0.011
Weibull($\beta = 2$)	Normal(0,1)	5	0.00	1.00	0.125	0.065	0.015
			0.00	0.80	0.109	0.055	0.011
			0.00	0.60	0.101	0.050	0.010
			0.20	1.00	0.103	0.052	0.011
			0.20	0.80	0.097	0.048	0.010
			0.40	1.00	0.101	0.050	0.010

Table G.2 Power of the Test: W_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Uniform(0,1)	5	0.00	1.00	0.114	0.061	0.014
			0.00	0.80	0.109	0.057	0.012
			0.00	0.60	0.108	0.053	0.010
			0.20	1.00	0.124	0.065	0.015
			0.20	0.80	0.105	0.053	0.011
			0.40	1.00	0.118	0.059	0.012
Weibull($\beta = 2$)	Beta(2,2)	5	0.00	1.00	0.107	0.054	0.011
			0.00	0.80	0.102	0.051	0.011
			0.00	0.60	0.101	0.050	0.010
			0.20	1.00	0.106	0.053	0.011
			0.20	0.80	0.098	0.048	0.010
			0.40	1.00	0.102	0.051	0.010
Weibull($\beta = 2$)	Beta(2,3)	5	0.00	1.00	0.096	0.045	0.009
			0.00	0.80	0.101	0.050	0.010
			0.00	0.60	0.098	0.049	0.009
			0.20	1.00	0.100	0.051	0.010
			0.20	0.80	0.099	0.049	0.010
			0.40	1.00	0.102	0.051	0.010
Weibull($\beta = 2$)	Chi-square(1)	5	0.00	1.00	0.328	0.232	0.106
			0.00	0.80	0.233	0.153	0.055
			0.00	0.60	0.170	0.096	0.025
			0.20	1.00	0.202	0.124	0.039
			0.20	0.80	0.134	0.072	0.017
			0.40	1.00	0.131	0.066	0.013
Weibull($\beta = 2$)	Chi-square(4)	5	0.00	1.00	0.132	0.070	0.015
			0.00	0.80	0.107	0.054	0.012
			0.00	0.60	0.101	0.050	0.009
			0.20	1.00	0.116	0.061	0.014
			0.20	0.80	0.099	0.050	0.010
			0.40	1.00	0.109	0.055	0.012

Table G.3 Power of the Test: W_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Log-normal(0,1)	5	0.00	1.00	0.273	0.184	0.073
			0.00	0.80	0.153	0.085	0.019
			0.00	0.60	0.108	0.054	0.010
			0.20	1.00	0.191	0.116	0.034
			0.20	0.80	0.110	0.056	0.011
			0.40	1.00	0.131	0.068	0.014
Weibull($\beta = 2$)	Log-logistic(0,1)	5	0.00	1.00	0.486	0.396	0.248
			0.00	0.80	0.253	0.168	0.060
			0.00	0.60	0.136	0.069	0.014
			0.20	1.00	0.350	0.265	0.135
			0.20	0.80	0.142	0.076	0.016
			0.40	1.00	0.216	0.131	0.038
Weibull($\beta = 2$)	Log-double exp(0,1)	5	0.00	1.00	0.392	0.312	0.187
			0.00	0.80	0.170	0.100	0.030
			0.00	0.60	0.109	0.053	0.010
			0.20	1.00	0.308	0.226	0.111
			0.20	0.80	0.118	0.061	0.013
			0.40	1.00	0.198	0.118	0.035
Weibull($\beta = 2$)	Log-Cauchy(0,1)	5	0.00	1.00	0.604	0.539	0.441
			0.00	0.80	0.283	0.206	0.111
			0.00	0.60	0.159	0.096	0.036
			0.20	1.00	0.516	0.455	0.357
			0.20	0.80	0.183	0.117	0.046
			0.40	1.00	0.414	0.340	0.239

Table G.4 Power of the Test: W_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Weibull($\beta = 1$)	15	0.00	1.00	0.573	0.464	0.270
			0.00	0.80	0.368	0.265	0.115
			0.00	0.60	0.240	0.155	0.052
			0.20	1.00	0.323	0.226	0.096
			0.20	0.80	0.171	0.103	0.030
			0.40	1.00	0.198	0.122	0.040
Weibull($\beta = 2$)	Weibull($\beta = 2$)	15	0.00	1.00	0.099	0.050	0.009
			0.00	0.80	0.099	0.051	0.010
			0.00	0.60	0.100	0.049	0.010
			0.20	1.00	0.100	0.051	0.010
			0.20	0.80	0.098	0.049	0.010
			0.40	1.00	0.097	0.048	0.010
Weibull($\beta = 2$)	Weibull($\beta = 3.5$)	15	0.00	1.00	0.258	0.166	0.058
			0.00	0.80	0.192	0.116	0.033
			0.00	0.60	0.152	0.085	0.021
			0.20	1.00	0.137	0.073	0.020
			0.20	0.80	0.115	0.059	0.012
			0.40	1.00	0.104	0.052	0.011
Weibull($\beta = 2$)	Gamma($\beta = 2$)	15	0.00	1.00	0.272	0.184	0.073
			0.00	0.80	0.160	0.095	0.026
			0.00	0.60	0.124	0.067	0.015
			0.20	1.00	0.185	0.110	0.035
			0.20	0.80	0.123	0.063	0.015
			0.40	1.00	0.147	0.083	0.022
Weibull($\beta = 2$)	Normal(0,1)	15	0.00	1.00	0.289	0.194	0.076
			0.00	0.80	0.228	0.148	0.049
			0.00	0.60	0.184	0.108	0.030
			0.20	1.00	0.127	0.070	0.018
			0.20	0.80	0.113	0.058	0.013
			0.40	1.00	0.103	0.052	0.012

Table G.5 Power of the Test: W_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Uniform(0,1)	15	0.00	1.00	0.192	0.111	0.031
			0.00	0.80	0.104	0.053	0.011
			0.00	0.60	0.122	0.066	0.016
			0.20	1.00	0.242	0.153	0.052
			0.20	0.80	0.111	0.055	0.012
			0.40	1.00	0.227	0.141	0.046
Weibull($\beta = 2$)	Beta(2,2)	15	0.00	1.00	0.193	0.113	0.031
			0.00	0.80	0.111	0.058	0.013
			0.00	0.60	0.104	0.053	0.011
			0.20	1.00	0.163	0.092	0.025
			0.20	0.80	0.106	0.053	0.011
			0.40	1.00	0.138	0.076	0.019
Weibull($\beta = 2$)	Beta(2,3)	15	0.00	1.00	0.102	0.051	0.011
			0.00	0.80	0.096	0.047	0.009
			0.00	0.60	0.098	0.051	0.010
			0.20	1.00	0.103	0.052	0.011
			0.20	0.80	0.099	0.050	0.010
			0.40	1.00	0.099	0.049	0.011
Weibull($\beta = 2$)	Chi-square(1)	15	0.00	1.00	0.886	0.828	0.673
			0.00	0.80	0.730	0.630	0.419
			0.00	0.60	0.548	0.435	0.236
			0.20	1.00	0.577	0.471	0.278
			0.20	0.80	0.333	0.232	0.096
			0.40	1.00	0.314	0.222	0.095
Weibull($\beta = 2$)	Chi-square(4)	15	0.00	1.00	0.270	0.184	0.072
			0.00	0.80	0.162	0.094	0.026
			0.00	0.60	0.122	0.065	0.015
			0.20	1.00	0.185	0.112	0.036
			0.20	0.80	0.121	0.064	0.014
			0.40	1.00	0.145	0.082	0.023

Table G.6 Power of the Test: W_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Log-normal(0,1)	15	0.00	1.00	0.755	0.679	0.519
			0.00	0.80	0.418	0.312	0.151
			0.00	0.60	0.222	0.141	0.046
			0.20	1.00	0.557	0.463	0.295
			0.20	0.80	0.232	0.149	0.050
			0.40	1.00	0.381	0.288	0.154
Weibull($\beta = 2$)	Log-logistic(0,1)	15	0.00	1.00	0.965	0.949	0.897
			0.00	0.80	0.754	0.670	0.483
			0.00	0.60	0.444	0.336	0.162
			0.20	1.00	0.874	0.830	0.727
			0.20	0.80	0.447	0.345	0.180
			0.40	1.00	0.715	0.642	0.501
Weibull($\beta = 2$)	Log-double exp(0,1)	15	0.00	1.00	0.879	0.844	0.762
			0.00	0.80	0.436	0.344	0.198
			0.00	0.60	0.170	0.104	0.032
			0.20	1.00	0.809	0.758	0.646
			0.20	0.80	0.305	0.219	0.100
			0.40	1.00	0.691	0.618	0.482
Weibull($\beta = 2$)	Log-Cauchy(0,1)	15	0.00	1.00	0.973	0.964	0.942
			0.00	0.80	0.674	0.599	0.451
			0.00	0.60	0.329	0.233	0.103
			0.20	1.00	0.944	0.929	0.896
			0.20	0.80	0.476	0.392	0.256
			0.40	1.00	0.908	0.884	0.836

Table G.7 Power of the Test: W_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Weibull($\beta = 1$)	25	0.00	1.00	0.808	0.727	0.531
			0.00	0.80	0.579	0.463	0.254
			0.00	0.60	0.395	0.279	0.120
			0.20	1.00	0.475	0.366	0.185
			0.20	0.80	0.239	0.152	0.053
			0.40	1.00	0.288	0.196	0.077
Weibull($\beta = 2$)	Weibull($\beta = 2$)	25	0.00	1.00	0.099	0.049	0.010
			0.00	0.80	0.099	0.049	0.010
			0.00	0.60	0.103	0.050	0.010
			0.20	1.00	0.097	0.047	0.009
			0.20	0.80	0.097	0.048	0.010
			0.40	1.00	0.099	0.049	0.010
Weibull($\beta = 2$)	Weibull($\beta = 3.5$)	25	0.00	1.00	0.394	0.281	0.121
			0.00	0.80	0.281	0.185	0.063
			0.00	0.60	0.215	0.129	0.039
			0.20	1.00	0.171	0.103	0.029
			0.20	0.80	0.127	0.068	0.016
			0.40	1.00	0.120	0.066	0.016
Weibull($\beta = 2$)	Gamma($\beta = 2$)	25	0.00	1.00	0.395	0.291	0.139
			0.00	0.80	0.220	0.139	0.047
			0.00	0.60	0.151	0.085	0.024
			0.20	1.00	0.254	0.166	0.060
			0.20	0.80	0.136	0.077	0.020
			0.40	1.00	0.184	0.110	0.033
Weibull($\beta = 2$)	Normal(0,1)	25	0.00	1.00	0.441	0.334	0.165
			0.00	0.80	0.349	0.249	0.107
			0.00	0.60	0.282	0.185	0.067
			0.20	1.00	0.155	0.088	0.024
			0.20	0.80	0.128	0.069	0.017
			0.40	1.00	0.112	0.058	0.014

Table G.8 Power of the Test: W_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Uniform(0,1)	25	0.00	1.00	0.295	0.182	0.058
			0.00	0.80	0.113	0.057	0.012
			0.00	0.60	0.150	0.084	0.022
			0.20	1.00	0.395	0.273	0.104
			0.20	0.80	0.117	0.062	0.015
			0.40	1.00	0.388	0.271	0.108
Weibull($\beta = 2$)	Beta(2,2)	25	0.00	1.00	0.296	0.189	0.061
			0.00	0.80	0.128	0.070	0.016
			0.00	0.60	0.105	0.052	0.010
			0.20	1.00	0.239	0.148	0.046
			0.20	0.80	0.118	0.061	0.013
			0.40	1.00	0.196	0.118	0.034
Weibull($\beta = 2$)	Beta(2,3)	25	0.00	1.00	0.116	0.059	0.013
			0.00	0.80	0.095	0.047	0.009
			0.00	0.60	0.098	0.048	0.010
			0.20	1.00	0.120	0.065	0.014
			0.20	0.80	0.099	0.049	0.010
			0.40	1.00	0.117	0.062	0.015
Weibull($\beta = 2$)	Chi-square(1)	25	0.00	1.00	0.989	0.977	0.932
			0.00	0.80	0.934	0.892	0.755
			0.00	0.60	0.817	0.727	0.522
			0.20	1.00	0.800	0.718	0.525
			0.20	0.80	0.511	0.395	0.210
			0.40	1.00	0.483	0.377	0.199
Weibull($\beta = 2$)	Chi-square(4)	25	0.00	1.00	0.397	0.293	0.141
			0.00	0.80	0.220	0.140	0.048
			0.00	0.60	0.153	0.086	0.023
			0.20	1.00	0.255	0.167	0.060
			0.20	0.80	0.140	0.079	0.021
			0.40	1.00	0.181	0.109	0.033

Table G.9 Power of the Test: W_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 2$)	Log-normal(0,1)	25	0.00	1.00	0.931	0.897	0.798
			0.00	0.80	0.635	0.529	0.325
			0.00	0.60	0.347	0.240	0.100
			0.20	1.00	0.777	0.701	0.530
			0.20	0.80	0.346	0.243	0.107
			0.40	1.00	0.584	0.488	0.316
Weibull($\beta = 2$)	Log-logistic(0,1)	25	0.00	1.00	0.999	0.997	0.992
			0.00	0.80	0.938	0.900	0.791
			0.00	0.60	0.692	0.585	0.369
			0.20	1.00	0.979	0.968	0.931
			0.20	0.80	0.670	0.571	0.377
			0.40	1.00	0.911	0.875	0.787
Weibull($\beta = 2$)	Log-double exp(0,1)	25	0.00	1.00	0.976	0.966	0.939
			0.00	0.80	0.612	0.524	0.356
			0.00	0.60	0.232	0.148	0.051
			0.20	1.00	0.947	0.926	0.872
			0.20	0.80	0.435	0.341	0.192
			0.40	1.00	0.896	0.860	0.768
Weibull($\beta = 2$)	Log-Cauchy(0,1)	25	0.00	1.00	0.998	0.997	0.995
			0.00	0.80	0.869	0.819	0.697
			0.00	0.60	0.501	0.390	0.205
			0.20	1.00	0.994	0.991	0.984
			0.20	0.80	0.647	0.570	0.426
			0.40	1.00	0.986	0.982	0.970

Appendix H. Power Study of W_s^2, H_0: Weibull($\beta = 3.5$)

Table H.1 Power of the Test: W_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Weibull($\beta = 1$)	5	0.00	1.00	0.260	0.170	0.060
			0.00	0.80	0.169	0.099	0.026
			0.00	0.60	0.124	0.063	0.012
			0.20	1.00	0.170	0.096	0.024
			0.20	0.80	0.116	0.058	0.012
			0.40	1.00	0.118	0.060	0.012
Weibull($\beta = 3.5$)	Weibull($\beta = 2$)	5	0.00	1.00	0.117	0.061	0.013
			0.00	0.80	0.101	0.052	0.011
			0.00	0.60	0.100	0.050	0.010
			0.20	1.00	0.110	0.058	0.012
			0.20	0.80	0.102	0.050	0.010
			0.40	1.00	0.103	0.053	0.011
Weibull($\beta = 3.5$)	Weibull($\beta = 3.5$)	5	0.00	1.00	0.099	0.049	0.009
			0.00	0.80	0.101	0.051	0.011
			0.00	0.60	0.101	0.051	0.010
			0.20	1.00	0.103	0.053	0.011
			0.20	0.80	0.100	0.049	0.011
			0.40	1.00	0.100	0.051	0.011
Weibull($\beta = 3.5$)	Gamma($\beta = 2$)	5	0.00	1.00	0.183	0.109	0.028
			0.00	0.80	0.126	0.067	0.014
			0.00	0.60	0.106	0.053	0.010
			0.20	1.00	0.132	0.071	0.015
			0.20	0.80	0.107	0.054	0.011
			0.40	1.00	0.110	0.057	0.012
Weibull($\beta = 3.5$)	Normal(0,1)	5	0.00	1.00	0.105	0.054	0.011
			0.00	0.80	0.098	0.050	0.010
			0.00	0.60	0.103	0.051	0.010
			0.20	1.00	0.102	0.052	0.011
			0.20	0.80	0.100	0.049	0.010
			0.40	1.00	0.101	0.050	0.010

Table H.2 Power of the Test: W_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Uniform(0,1)	5	0.00	1.00	0.103	0.054	0.011
			0.00	0.80	0.115	0.062	0.013
			0.00	0.60	0.115	0.059	0.011
			0.20	1.00	0.114	0.062	0.014
			0.20	0.80	0.106	0.053	0.011
			0.40	1.00	0.111	0.057	0.012
Weibull($\beta = 3.5$)	Beta(2,2)	5	0.00	1.00	0.094	0.047	0.010
			0.00	0.80	0.101	0.051	0.011
			0.00	0.60	0.103	0.052	0.010
			0.20	1.00	0.100	0.051	0.010
			0.20	0.80	0.103	0.051	0.010
			0.40	1.00	0.100	0.050	0.010
Weibull($\beta = 3.5$)	Beta(2,3)	5	0.00	1.00	0.099	0.050	0.011
			0.00	0.80	0.102	0.052	0.011
			0.00	0.60	0.105	0.052	0.010
			0.20	1.00	0.100	0.050	0.011
			0.20	0.80	0.101	0.051	0.010
			0.40	1.00	0.101	0.051	0.011
Weibull($\beta = 3.5$)	Chi-square(1)	5	0.00	1.00	0.407	0.302	0.149
			0.00	0.80	0.272	0.190	0.075
			0.00	0.60	0.190	0.110	0.029
			0.20	1.00	0.238	0.153	0.050
			0.20	0.80	0.146	0.077	0.018
			0.40	1.00	0.140	0.073	0.016
Weibull($\beta = 3.5$)	Chi-square(4)	5	0.00	1.00	0.182	0.109	0.029
			0.00	0.80	0.123	0.065	0.014
			0.00	0.60	0.105	0.052	0.009
			0.20	1.00	0.135	0.074	0.016
			0.20	0.80	0.107	0.053	0.011
			0.40	1.00	0.109	0.057	0.011

Table H.3 Power of the Test: W_s^2 Sample Size = 5

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Log-normal(0,1)	5	0.00	1.00	0.350	0.254	0.112
			0.00	0.80	0.185	0.110	0.028
			0.00	0.60	0.119	0.060	0.011
			0.20	1.00	0.228	0.145	0.044
			0.20	0.80	0.119	0.060	0.012
			0.40	1.00	0.139	0.073	0.015
Weibull($\beta = 3.5$)	Log-logistic(0,1)	5	0.00	1.00	0.555	0.463	0.307
			0.00	0.80	0.298	0.207	0.080
			0.00	0.60	0.153	0.079	0.017
			0.20	1.00	0.390	0.302	0.158
			0.20	0.80	0.159	0.084	0.018
			0.40	1.00	0.226	0.141	0.043
Weibull($\beta = 3.5$)	Log-double exp(0,1)	5	0.00	1.00	0.463	0.379	0.239
			0.00	0.80	0.205	0.128	0.041
			0.00	0.60	0.117	0.059	0.012
			0.20	1.00	0.347	0.261	0.129
			0.20	0.80	0.129	0.066	0.014
			0.40	1.00	0.214	0.130	0.038
Weibull($\beta = 3.5$)	Log-Cauchy(0,1)	5	0.00	1.00	0.650	0.586	0.481
			0.00	0.80	0.315	0.236	0.127
			0.00	0.60	0.174	0.108	0.039
			0.20	1.00	0.547	0.484	0.377
			0.20	0.80	0.194	0.122	0.047
			0.40	1.00	0.424	0.352	0.248

Table H.4 Power of the Test: W_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Weibull($\beta = 1$)	15	0.00	1.00	0.795	0.709	0.504
			0.00	0.80	0.552	0.433	0.233
			0.00	0.60	0.352	0.243	0.099
			0.20	1.00	0.485	0.379	0.196
			0.20	0.80	0.235	0.151	0.049
			0.40	1.00	0.280	0.191	0.070
Weibull($\beta = 3.5$)	Weibull($\beta = 2$)	15	0.00	1.00	0.223	0.139	0.044
			0.00	0.80	0.152	0.087	0.024
			0.00	0.60	0.122	0.065	0.015
			0.20	1.00	0.152	0.086	0.021
			0.20	0.80	0.111	0.058	0.012
			0.40	1.00	0.121	0.064	0.014
Weibull($\beta = 3.5$)	Weibull($\beta = 3.5$)	15	0.00	1.00	0.100	0.050	0.009
			0.00	0.80	0.103	0.051	0.010
			0.00	0.60	0.102	0.050	0.010
			0.20	1.00	0.103	0.052	0.011
			0.20	0.80	0.101	0.051	0.009
			0.40	1.00	0.100	0.051	0.011
Weibull($\beta = 3.5$)	Gamma($\beta = 2$)	15	0.00	1.00	0.551	0.441	0.238
			0.00	0.80	0.316	0.216	0.086
			0.00	0.60	0.193	0.116	0.034
			0.20	1.00	0.328	0.229	0.094
			0.20	0.80	0.160	0.095	0.025
			0.40	1.00	0.208	0.131	0.040
Weibull($\beta = 3.5$)	Normal(0,1)	15	0.00	1.00	0.120	0.063	0.014
			0.00	0.80	0.113	0.059	0.014
			0.00	0.60	0.110	0.056	0.013
			0.20	1.00	0.110	0.055	0.011
			0.20	0.80	0.100	0.051	0.010
			0.40	1.00	0.107	0.054	0.010

Table H.5 Power of the Test: W_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Uniform(0,1)	15	0.00	1.00	0.111	0.052	0.009
			0.00	0.80	0.159	0.090	0.025
			0.00	0.60	0.175	0.102	0.030
			0.20	1.00	0.153	0.087	0.022
			0.20	0.80	0.098	0.049	0.010
			0.40	1.00	0.176	0.105	0.029
Weibull($\beta = 3.5$)	Beta(2,2)	15	0.00	1.00	0.075	0.034	0.005
			0.00	0.80	0.101	0.051	0.011
			0.00	0.60	0.106	0.054	0.012
			0.20	1.00	0.095	0.048	0.010
			0.20	0.80	0.095	0.046	0.008
			0.40	1.00	0.100	0.052	0.011
Weibull($\beta = 3.5$)	Beta(2,3)	15	0.00	1.00	0.115	0.059	0.011
			0.00	0.80	0.137	0.075	0.019
			0.00	0.60	0.127	0.068	0.016
			0.20	1.00	0.089	0.043	0.008
			0.20	0.80	0.102	0.052	0.011
			0.40	1.00	0.090	0.044	0.007
Weibull($\beta = 3.5$)	Chi-square(1)	15	0.00	1.00	0.955	0.925	0.819
			0.00	0.80	0.832	0.752	0.555
			0.00	0.60	0.649	0.535	0.319
			0.20	1.00	0.717	0.624	0.421
			0.20	0.80	0.412	0.304	0.138
			0.40	1.00	0.418	0.314	0.150
Weibull($\beta = 3.5$)	Chi-square(4)	15	0.00	1.00	0.551	0.439	0.238
			0.00	0.80	0.314	0.214	0.084
			0.00	0.60	0.194	0.116	0.035
			0.20	1.00	0.325	0.229	0.094
			0.20	0.80	0.163	0.097	0.027
			0.40	1.00	0.207	0.130	0.040

Table H.6 Power of the Test: W_s^2 Sample Size = 15

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Log-normal(0,1)	15	0.00	1.00	0.896	0.847	0.714
			0.00	0.80	0.605	0.494	0.287
			0.00	0.60	0.332	0.230	0.092
			0.20	1.00	0.699	0.612	0.432
			0.20	0.80	0.308	0.211	0.081
			0.40	1.00	0.474	0.377	0.214
Weibull($\beta = 3.5$)	Log-logistic(0,1)	15	0.00	1.00	0.988	0.979	0.948
			0.00	0.80	0.857	0.790	0.623
			0.00	0.60	0.561	0.446	0.243
			0.20	1.00	0.925	0.892	0.804
			0.20	0.80	0.531	0.427	0.239
			0.40	1.00	0.777	0.712	0.570
Weibull($\beta = 3.5$)	Log-double exp(0,1)	15	0.00	1.00	0.947	0.926	0.866
			0.00	0.80	0.595	0.500	0.328
			0.00	0.60	0.254	0.165	0.062
			0.20	1.00	0.874	0.832	0.734
			0.20	0.80	0.375	0.281	0.138
			0.40	1.00	0.752	0.688	0.542
Weibull($\beta = 3.5$)	Log-Cauchy(0,1)	15	0.00	1.00	0.988	0.983	0.968
			0.00	0.80	0.790	0.721	0.571
			0.00	0.60	0.429	0.319	0.158
			0.20	1.00	0.964	0.952	0.922
			0.20	0.80	0.540	0.456	0.304
			0.40	1.00	0.928	0.909	0.860

Table H.7 Power of the Test: W_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Weibull($\beta = 1$)	25	0.00	1.00	0.962	0.932	0.838
			0.00	0.80	0.807	0.715	0.502
			0.00	0.60	0.591	0.469	0.248
			0.20	1.00	0.710	0.613	0.404
			0.20	0.80	0.360	0.257	0.105
			0.40	1.00	0.433	0.326	0.161
Weibull($\beta = 3.5$)	Weibull($\beta = 2$)	25	0.00	1.00	0.347	0.243	0.100
			0.00	0.80	0.220	0.137	0.043
			0.00	0.60	0.158	0.091	0.024
			0.20	1.00	0.188	0.111	0.034
			0.20	0.80	0.125	0.069	0.017
			0.40	1.00	0.139	0.077	0.019
Weibull($\beta = 3.5$)	Weibull($\beta = 3.5$)	25	0.00	1.00	0.102	0.052	0.011
			0.00	0.80	0.100	0.050	0.011
			0.00	0.60	0.103	0.051	0.011
			0.20	1.00	0.100	0.049	0.010
			0.20	0.80	0.099	0.052	0.010
			0.40	1.00	0.097	0.049	0.010
Weibull($\beta = 3.5$)	Gamma($\beta = 2$)	25	0.00	1.00	0.795	0.707	0.512
			0.00	0.80	0.501	0.386	0.193
			0.00	0.60	0.301	0.203	0.075
			0.20	1.00	0.495	0.384	0.197
			0.20	0.80	0.223	0.142	0.044
			0.40	1.00	0.307	0.211	0.086
Weibull($\beta = 3.5$)	Normal(0,1)	25	0.00	1.00	0.130	0.069	0.017
			0.00	0.80	0.123	0.065	0.016
			0.00	0.60	0.121	0.064	0.015
			0.20	1.00	0.110	0.057	0.012
			0.20	0.80	0.103	0.053	0.011
			0.40	1.00	0.109	0.057	0.013

Table H.8 Power of the Test: W_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Uniform(0,1)	25	0.00	1.00	0.175	0.087	0.017
			0.00	0.80	0.234	0.144	0.046
			0.00	0.60	0.280	0.182	0.064
			0.20	1.00	0.218	0.130	0.039
			0.20	0.80	0.096	0.050	0.010
			0.40	1.00	0.275	0.178	0.062
Weibull($\beta = 3.5$)	Beta(2,2)	25	0.00	1.00	0.080	0.036	0.005
			0.00	0.80	0.114	0.060	0.014
			0.00	0.60	0.124	0.068	0.016
			0.20	1.00	0.105	0.053	0.012
			0.20	0.80	0.093	0.047	0.009
			0.40	1.00	0.121	0.065	0.016
Weibull($\beta = 3.5$)	Beta(2,3)	25	0.00	1.00	0.161	0.088	0.022
			0.00	0.80	0.190	0.114	0.034
			0.00	0.60	0.161	0.092	0.025
			0.20	1.00	0.088	0.041	0.007
			0.20	0.80	0.109	0.057	0.012
			0.40	1.00	0.086	0.041	0.008
Weibull($\beta = 3.5$)	Chi-square(1)	25	0.00	1.00	0.999	0.997	0.985
			0.00	0.80	0.979	0.958	0.883
			0.00	0.60	0.897	0.835	0.658
			0.20	1.00	0.916	0.868	0.729
			0.20	0.80	0.636	0.529	0.308
			0.40	1.00	0.629	0.526	0.329
Weibull($\beta = 3.5$)	Chi-square(4)	25	0.00	1.00	0.793	0.708	0.510
			0.00	0.80	0.498	0.383	0.191
			0.00	0.60	0.301	0.202	0.072
			0.20	1.00	0.494	0.382	0.196
			0.20	0.80	0.224	0.144	0.046
			0.40	1.00	0.303	0.208	0.085

Table H.9 Power of the Test: W_s^2 Sample Size = 25

Null Distribution	Alternative Distribution	Sample size	Cnsr level L	Cnsr level R	Significance level α 0.10	0.05	0.01
Weibull($\beta = 3.5$)	Log-normal(0,1)	25	0.00	1.00	0.990	0.981	0.949
			0.00	0.80	0.848	0.774	0.588
			0.00	0.60	0.552	0.433	0.223
			0.20	1.00	0.901	0.854	0.727
			0.20	0.80	0.479	0.371	0.183
			0.40	1.00	0.708	0.623	0.449
Weibull($\beta = 3.5$)	Log-logistic(0,1)	25	0.00	1.00	1.000	1.000	0.999
			0.00	0.80	0.982	0.966	0.906
			0.00	0.60	0.825	0.741	0.533
			0.20	1.00	0.993	0.987	0.968
			0.20	0.80	0.770	0.685	0.487
			0.40	1.00	0.945	0.920	0.852
Weibull($\beta = 3.5$)	Log-double exp(0,1)	25	0.00	1.00	0.996	0.993	0.985
			0.00	0.80	0.811	0.741	0.584
			0.00	0.60	0.391	0.285	0.128
			0.20	1.00	0.979	0.969	0.938
			0.20	0.80	0.554	0.462	0.282
			0.40	1.00	0.937	0.910	0.838
Weibull($\beta = 3.5$)	Log-Cauchy(0,1)	25	0.00	1.00	0.999	0.999	0.999
			0.00	0.80	0.946	0.917	0.831
			0.00	0.60	0.665	0.557	0.339
			0.20	1.00	0.997	0.996	0.992
			0.20	0.80	0.737	0.667	0.515
			0.40	1.00	0.992	0.989	0.980

Vita

Eric William Frisco was born on 30 March 1970 in Chestnut Hill, Pennsylvania. He graduated from Central Bucks High School East in Buckingham, Pennsylvania in 1988 and began undergraduate studies at Embry-Riddle Aeronautical University in Daytona Beach, Florida. He received a B.S. in Aerospace Engineering on 12 December 1992. In the following year, he pursued a commission in the United States Air Force through the Air Force Officer Training School (AF OTS) program. He graduated from AF OTS, Maxwell AFB, Alabama on 15 March 1994 as a distinguished graduate and received a commission in the United States Air Force. After commissioning, he served as a Weapon Systems Analyst in the Air Force Ballistic Missile Defense Program Office, Systems Engineering and Integration Division at Los Angeles AFB, California. During this assignment, he was married to Cecilia Pik-Hung Ip on 12 May 1996 at Wayfarers Chapel in Rancho Palos Verdes, California. In August 1996, he entered the Graduate Operations Research program in the School of Engineering, Air Force Institute of Technology, Wright-Patterson AFB, Ohio.

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13. ABSTRACT *(Maximum 200 words)*
Two new goodness-of-fit tests are developed for the three-parameter Weibull distribution with known shape parameter. These procedures eliminate the need for estimating the unknown location and scale parameters prior to initiating the tests and are easily adapted for censored data. This is accomplished by employing the Anderson-Darling and Cramer-von Mises statistics based on the normalized spacings of the sample data. Critical values of the new tests are obtained for common significance levels by large Monte Carlo simulations for shapes 0.5(0.5)4.0 and sample sizes 5(5)40 with up to 40% censoring (Type II) from the left and/or right. An extensive Monte Carlo power study is also conducted to compare the two tests with each other and with their prominent competitors. The competitors include another spacings test, Z*, and the modified Kolmogorov-Smirnov, Cramer-von Mises, and Anderson-Darling EDF tests. Results show the Anderson-Darling spacings test is the preferred test for the three-parameter Weibull distribution with known shape parameter.

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